



This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

- + *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
- + *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
- + *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
- + *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at <http://books.google.com/>

FIRST PRINCIPLES OF

ALGEBRA

SLAUGHT AND LENNES

Educ T 129.12.797

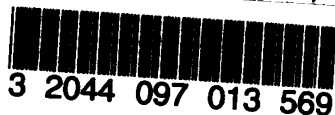
Edwin Harris
#10



Harvard College Library

FROM

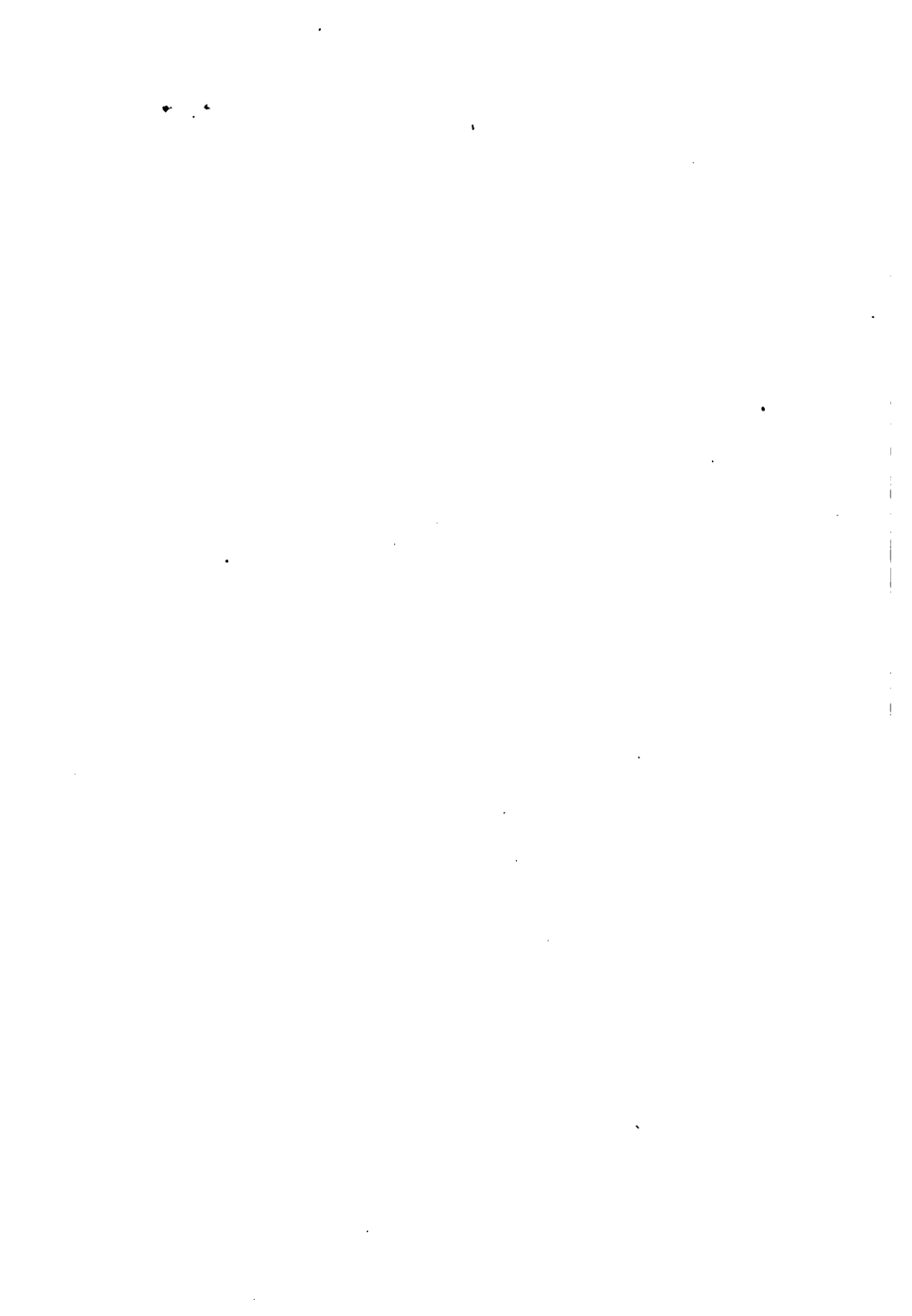
Fulmer Mood



Gertrude Harris.

10.

8442



FIRST PRINCIPLES OF ALGEBRA

Elementary Course

BY

H. E. SLAUGHT, PH.D., Sc.D.

ASSOCIATE PROFESSOR OF MATHEMATICS IN THE UNIVERSITY
OF CHICAGO

AND

N. J. LENNES, PH.D.

INSTRUCTOR IN MATHEMATICS IN COLUMBIA UNIVERSITY



ALLYN AND BACON

Boston and Chicago

Edue T 129.12.797

✓



Fulmer Mowbray

COPYRIGHT, 1912,
BY H. E. SLAUGHT
AND N. J. LENNES.

FDT

Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

PREFACE

IN writing the First Principles of Algebra the authors have had constantly before them two chief aims:

(1) To provide a gradual and natural introduction to the symbols and processes of algebra.

(2) To give vital purpose to the study of algebra by using it to do interesting and valuable things.

Each of these aims leads to the same order of topics, which, however, differs somewhat from the conventional order.

If it is admitted that there should be a gradually increasing complexity of forms to be manipulated, it follows that factoring and complicated work in fractions have no proper place in the first half year. This book is arranged so that factoring may begin with the second semester and complicated fractions may come still later. Simple fractions are treated in Chapter V.

The pupil is introduced to the algebraic notation by recalling and stating in terms of letters certain rules of arithmetic with which he is already familiar. The simplicity of the algebraic formulas, compared with the arithmetical statement of rules known to the pupil, cannot fail to impress him with the usefulness and power of the subject which he is about to study. This impression will be deepened when, in Chapter VI, rules which caused considerable trouble in arithmetic are derived with the utmost ease by algebraic processes.

The study of equations and their uses is naturally the main topic of the Elementary Course. This topic is therefore developed early, simple simultaneous equations being completed in Part One, which is intended to be covered in the first half year. It is recognized that abstract equations will appear of little or

no value to the pupil unless he finds uses for them. Hence frequent lists of problems are provided for translation into equations and for solution.

Many of these problems involve valuable mathematical concepts, such as the dimensions and areas of rectangles and triangles, the constitution of the decimal system (digit problems), the fundamental relations connecting velocity, distance, and time, properties of the lever, and so forth. There are also numerous artificial problems involving relations imposed upon abstract numbers or upon interesting informational data.

In this way, during the first half year, *algebra is made to appeal to the higher and more useful types of interest*, and not merely to the instinct for solving puzzles, which must be the case if the greater part of this time is spent on factoring and in manipulating complicated fractions.

The work of the second half year is intended to begin with special products and factors, long division being introduced in this connection. Factoring is at once applied to the solution of quadratic equations and problems depending on them.

In the chapters on radicals and quadratic equations, simple facts from geometry, which the pupil learned in arithmetic, afford the basis for a large number of problems that are of legitimate interest both on their own account, and because of the valuable geometrical relations involved.

The more complicated algebraic fractions, which serve little purpose in a first course except to afford drill in algebraic manipulation, are placed at about the middle of Part Two and are followed by chapters on Ratio, Variation, and Proportion, Equations involving Algebraic Fractions, and a General Review.

For the development of skill in algebraic manipulation it is not sufficient to solve a certain number of exercises when an operation is first introduced. To fix each operation in

the learner's mind, there must be recurring drills extending over a considerable period of time. These are amply provided for in this book. The fundamental operations on integral and simple fractional expressions, the solution of simple equations, and the representation of given conditions in algebraic symbols are constantly reviewed in the numerous lists of "drill exercises," many of which may be solved mentally. Factoring is practised almost daily throughout the second half year.

The principles of algebra used in the Elementary Course are enunciated in a small number of short rules — eighteen in all. The purpose of these rules is to furnish, in simple form, a codification of those operations of algebra which require special emphasis. Such a codification has several important advantages:

By constant reference to these few fundamental statements they become an organic, and hence a permanent, part of the learner's mental equipment.

By their systematic use he is made to realize that the processes of algebra, which seem so multifarious and heterogeneous, are, in reality, few and simple.

Such a body of principles furnishes a ready means for the correction of erroneous notions, a constant incitement to effective review, and a definite basis upon which to proceed at each stage of progress.

The authors gratefully acknowledge the receipt of many helpful suggestions from teachers who have used their High School Algebra.

H. E. SLAUGHT.
N. J. LENNES.

CHICAGO AND NEW YORK,
April, 1912.

TABLE OF CONTENTS

PART ONE

CHAPTER	PAGE
I. Introduction to Algebra	1-19
II. Equations and Problems	20-35
III. Positive and Negative Numbers	36-55
IV. Polynomials	56-90
V. Simple Fractions	81-91
VI. Literal Equations and their Uses	92-101
VII. Graphic Representation	102-115
VIII. Simultaneous Equations	116-133

PART TWO

IX. Products, Quotients, and Factors	134-165
X. Equations Solved by Factoring	166-175
XI. Square Roots and Radicals	176-198
XII. Quadratic Equations	199-214
XIII. Algebraic Fractions	215-235
XIV. Ratio, Variation, and Proportion	236-244
XV. Equations involving Fractions	245-254
XVI. General Review	255-276
INDEX	277

LIST OF PRINCIPLES

NUMBERS	PAGE
I. Addition and Subtraction of Numbers having a Common Factor	6
II. Multiplication of the Sum or Difference of Two Numbers	9
III. Division of the Sum or Difference of Two Numbers .	10
IV. Multiplication of the Product of Several Factors . .	12
V. Division of the Product of Several Factors . . .	13
VI. Deducing One Equation from Another	24
VII. Addition of Signed Numbers	39
VIII. Subtraction of Signed Numbers	43
IX. Multiplication of Signed Numbers	47
X. Division of Signed Numbers	49
XI. Arrangement and Grouping of Terms	57
XII. Removal and Insertion of Parentheses	62
XIII. Arrangement and Grouping of Factors	67
XIV. Multiplication of Polynomials	70
XV. Multiplying or Dividing the Terms of a Fraction . .	82
XVI. Products of Powers of the Same Base	136
XVII. Quotients of Powers of the Same Base	138
XVIII. Square Roots of Monomials	177

FIRST PRINCIPLES OF ALGEBRA

ELEMENTARY COURSE

PART ONE

CHAPTER I

INTRODUCTION TO ALGEBRA

LETTERS USED TO REPRESENT NUMBERS

1. **Algebra**, like arithmetic, deals with numbers. In arithmetic numbers are represented by means of the Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. In algebra letters, as well as these numerals, are used to represent numbers.

2. One advantage in using letters to represent numbers is in abbreviating the rules of arithmetic.

Examples. What is the area of a blackboard whose length is 12 feet and whose width is 3 feet? What is the area of a city lot which is 25 feet wide and 120 feet deep?

In arithmetic we multiply the length by the width and the product is the area. In algebra we abbreviate by letting l represent the *length*, w the *width*, and a the *area*.

The rule then stands as follows:

$$l \times w = a$$

This is called a **formula** and covers all possible cases.

It is to be noted that l and w represent the number of units in the length and width respectively and a the number of square units in the area.

3. The signs of operation, $+$, $-$, \times , \div , and the sign of equality, $=$, are used in algebra with the same meaning that they have in arithmetic. However, the sign of multiplication is usually omitted between letters, and between an Arabic figure and a letter; but a sign omitted between two Arabic figures always means addition. Multiplication is also indicated by a point written above the line.

Thus, $2 \times l \times w$ may be written $2 \cdot l \cdot w$ or $2lw$.

But 25 means $20 + 5$, not $2 \cdot 5$.

Division may be indicated by a fraction, as in arithmetic.

Thus, $2 \div 3$ is written $\frac{2}{3}$ and $a \div b$ is written $\frac{a}{b}$.

Historical Note. The Arabs brought our present system of numerals to Europe when they invaded Spain in the eighth century of our era. It is now known that these numerals (including the zero) are really of Hindu origin, and that they were invented between the years 200 A.D. and 650 A.D. Of all mathematical inventions no one has contributed more to general intelligence or to practical welfare than this one. To appreciate this, one has only to try to multiply two numbers such as 589 and 642 when expressed in the Roman notation; that is, to multiply DLXXXIX by DCXLII.

It was about the year 1500 A.D. that our present symbols indicating addition and subtraction first appeared in a book by a German named Johann Widemann. The sign \times for multiplication was first used about 50 years later by an Englishman, William Oughtred. About the same time the sign $=$ was first used by Recorde, also an Englishman; but the sign \div for division does not appear until 1659 when it was used by a German, J. H. Rohn.

4. In the following exercises, note the simplicity with which the rules of arithmetic may be stated by means of letters. The systematic use of letters to represent numbers is one of the chief points of difference between algebra and arithmetic.

EXERCISES

① A box is 6 inches long, 4 inches wide, and 3 inches high. How many cubic inches does it hold?

② How many cubic feet of air in a schoolroom which is 35 feet long, 25 feet wide, and 15 feet high?

3. Given the length, width, and height of a rectangular solid, how do you find its volume? State this rule as a formula, using l , w , h , and v for the *number* of units in the length, width, height, and volume, respectively.

4. By a rule of arithmetic the product of $\frac{2}{3}$ and $\frac{5}{7}$ is $\frac{2 \cdot 5}{3 \cdot 7}$. What is the rule for finding the product of any two fractions? This rule may be stated as a formula as follows:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

in which $\frac{a}{b}$ and $\frac{c}{d}$ are any two fractions.

5. By a rule of arithmetic $\frac{3}{4} \div \frac{5}{7} = \frac{3}{4} \times \frac{7}{5} = \frac{3 \cdot 7}{4 \cdot 5}$. What is the rule for dividing one fraction by another? Using $\frac{a}{b}$ and $\frac{c}{d}$ to represent any two fractions, state this rule as a formula.

6. By arithmetic 8 per cent of 90 means $90 \times .08 = 7.2$, in which 90 is called the *base*, .08 the *rate*, and 7.2 the *percentage*.

In like manner find 6 per cent of 125; 5 per cent of 350; 3 per cent of 80; and 10 per cent of 4.9. In each case how is the percentage found? State the rule.

State this rule as a formula, letting b represent the base, r the rate and p the percentage.

7. By a rule of arithmetic the interest on \$ 500 for 3 years at 6 per cent is $500 \times .06 \times 3 = 90$, in which 500 is the *principal*, .06 the *rate*, and 3 the *time*.

8. Find the interest on \$ 900 for 7 years at 4 per cent, and on \$ 1250 for 5 years at 5 per cent.

9. Given any principal, rate, and time, how do you find the interest?

State this rule as a formula, using p for principal, r for rate, t for time and i for interest.

ALGEBRAIC OPERATIONS

5. In algebra numbers are added, subtracted, multiplied, and divided as in arithmetic. There are, however, certain ways of carrying out these operations which are essential in algebra though not used extensively in arithmetic. Some of these we now proceed to study.

6. **Definitions.** Any combination of Arabic numerals, letters, and signs of operation, used for the purpose of representing numbers, is called a **number expression**.

E.g. 38, $18r$, $5n + 8n$, $\frac{3a-2}{b}$ are number expressions.

The expression $\frac{3a-2}{b}$ is said to be written in symbols, that is, by means of numerals, letters, and signs of operation.

This number expression written in words would be "three times the number a diminished by 2 and the result divided by b ."

EXERCISES

1. If a and b are numbers, express in symbols their sum and also their product.

2. If m and n are numbers, write in symbols m divided by n ; also m minus n .

3. If p and q are numbers, write the sum of 5 times p and 3 times q .

4. If a , b , and c are numbers, write in symbols that a multiplied by b equals c ; also that c divided by a equals b .

If $a = 3$, $b = 5$, $c = 7$, find the value of each of the following number expressions:

5. $a + b + c$.

10. $bc - 3a$.

13. $3abc - 2b + 3c$.

6. $a + b - c$.

11. $\frac{b+c}{a}$.

14. $\frac{2a+3b}{c} + 4b$.

7. $2a - b + c$.

8. $ab - 2c$.

12. $\frac{4a-b}{c}$.

15. $\frac{3c-2a}{3b}$.

9. $16a - bc$.

7. **Definition.** If a number is the product of two or more numbers, these are together called the **cofactors** of the given number, and any one of them is called a **factor**. ✓

E.g. 3 and 4 are cofactors of 12, as are also 2 and 6, 1 and 12. 1, 2, 3, 4, 6, and 12 are factors of 12. a , b , c are factors of abc .

8. **Definition.** If a number is the product of two factors, then either of these factors is called the **coefficient** of the other in that product. ✓

E.g. In $2 \cdot 3$, 2 is the coefficient of 3, and 3 is the coefficient of 2. In $9rt$, 9 is the coefficient of rt , r is the coefficient of $9t$, and t is the coefficient of $9r$.

In such expressions as $9rt$ the factor represented by Arabic figures is usually regarded as the coefficient.

ADDITION AND SUBTRACTION OF NUMBERS HAVING A COMMON FACTOR

9. It is sometimes necessary to add number expressions like $5n$ and $8n$ without first assigning a definite value to n .

It seems clear that 5 times any number and 8 times the same number make 13 times that number. That is,

$$5n + 8n = 13n.$$

Determine whether $5n + 8n = 13n$ when $n = 3$; also when $n = 9$; when $n = 12$. Test this for still other values of n .

Note that the sum of $5 \cdot 3$ and $8 \cdot 3$ may be found by adding the coefficients of the common factor 3, while the sum of $5n$ and $8n$ must be found in this way.) ?

In this manner perform the following additions:

- | | |
|---------------------------|-------------------------|
| 1. $8x + 7x + 16x + 2x$. | 4. $7b + 8b + b + 6b$. |
| 2. $13n + 8n + 7n + 9n$. | 5. $8t + 7t + 5t$. |
| 3. $3a + a + 2a + 6a$. | 6. $3r + 5r + 11r$. |

Test the correctness of the result in each of the above by letting $x = 2$, $n = 1$, $a = 4$, $b = 3$, $t = 6$, and $r = 5$.

10. Numbers having a common factor may be subtracted in a similar manner.

Thus, from $64 = 8 \cdot 8$ subtract $48 = 6 \cdot 8$ Remainder $16 = 2 \cdot 8$	From $84 = 12 \cdot 7$ subtract $49 = 7 \cdot 7$ $35 = 5 \cdot 7$	From $17n$ subtract $6n$ $11n$
----------------------------------------------------------------------------------------	-------------------------------------------------------------------------	--------------------------------------

In this way perform the following subtractions:

- | | | |
|--------------------------------|------------------|-------------------|
| 1. $8 \cdot 7 - 3 \cdot 7$. | 5. $10b - 4b$. | 9. $14a - 8a$. |
| 2. $6 \cdot 99 - 5 \cdot 99$. | 6. $7a - 4a$. | 10. $12b - 9b$. |
| 3. $6n - 2n$. | 7. $23x - 16x$. | 11. $8t - 2t$. |
| 4. $8a - 3a$. | 8. $15n - 3n$. | 12. $19r - 11r$. |

Test these results by giving the same values to the letters as were used on page 5. Try also other values.

It is evident that any two numbers having a common factor may be added or subtracted in this manner.

These examples illustrate

Principle I

11. **Rule.** *To find the sum or difference of two numbers having a common factor, add or subtract the coefficients of the common factor and multiply the result by the common factor.*

12. **Definition.** The substitution of special values for letters in a number expression in order to test the correctness of an operation is called a **check**.

EXERCISES.

Perform the following indicated operations and check the results in the first eight by assigning values to the letters:

- | | |
|-----------------------------|--------------------------|
| 1. $68t - 11t$. | 5. $20n - 6n + 2n$. |
| 2. $15n + 25n - 18n$. | 6. $5t + 20t - 3t$. |
| 3. $70x - 15x + 7x - 23x$. | 7. $8s - 3s + 20s$. |
| 4. $18k - 3k - 2k + 6k$. | 8. $6a - 4a + 3a - 2a$. |

- | | |
|------------------------------------|-----------------------------|
| 9. $7ab - 3ab + 2ab.$ | 17. $32ac - 17ac + 2ac.$ |
| 10. $11rs - 2rs - 5rs.$ | 18. $91a - 81a + 2a.$ |
| 11. $34xy - 18xy - 4xy.$ | 19. $16x + 24x + 8x - 40x.$ |
| 12. $12mn - 6mn - 3mn. \checkmark$ | 20. $5y + 31y - 9y - 21y.$ |
| 13. $17st + 3st - 12st.$ | 21. $63c - 47c - 8c + 7c.$ |
| 14. $12abc - 2abc - 6abc.$ | 22. $16t - 11t - 2t + 3t.$ |
| 15. $42xy + 6xy - 35xy.$ | 23. $12xy - 9xy + 8xy.$ |
| 16. $29rst - 18rst - 6rst.$ | 24. $39ab - 27ab - 8ab.$ |

13. Definitions. Two number expressions representing the same number, when connected by the sign $=$, form an *equality*. *Equality*

The expressions thus connected are called the **members** of the equality and are distinguished as the **right** and **left** members.

Equalities, such as $5n + 8n = 13n$, in which the letters may be *any numbers whatever*, are called **identities**. Not all equalities are of this kind. For example, $n + 3 = 5$ is true only when $n = 2$.

When it is desired to emphasize that an equality is an identity, the sign \equiv is used. That is, $3a + 5a \equiv 8a$.

14. Definitions. A parenthesis is used to indicate that some operation is to be extended over the whole number expression inclosed by it. Thus $2(x + y)$ means that the *sum* of x and y is to be multiplied by 2, while $2x + y$ means that x alone is to be multiplied by 2.

Instead of a parenthesis a **bracket** $[]$, or a **brace** $\{ \}$, may be used with the same meaning. Any such symbols are called **symbols of aggregation**.

E.g. $2(x + y)$, $2[x + y]$ or $2\{x + y\}$ all mean the same thing.

Historical Note. The parenthesis $()$ was first used with its present meaning by an Englishman, A. Girard, in a book on "Arithmetic," published in the year 1629. The bracket and brace are of later origin, as is also the sign \equiv to denote identity.

MULTIPLICATION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

15. The sum of two or more numbers may be multiplied by another number in two ways, as shown in the following examples:

$$(1) \quad 4(2+7) = 4 \cdot 9 = 36,$$

$$\text{or} \quad 4(2+7) = (4 \cdot 2) + (4 \cdot 7) = 8 + 28 = 36.$$

$$(2) \quad 3(3+8+9) = (3 \cdot 20) = 60,$$

$$\text{or} \quad 3(3+8+9) = (3 \cdot 3) + (3 \cdot 8) + (3 \cdot 9) = 9 + 24 + 27 = 60.$$

In each case the same result is obtained *whether we first add the numbers in the parenthesis and then multiply the sum, or first multiply these numbers one by one and then add the products.*

In case the numbers are represented by letters, the second process only is available.

$$\text{E.g.} \quad 3(a+b) = 3a + 3b \text{ and } m(r+s) = mr + ms.$$

Multiply each of the following in two ways where possible:

$$1. \quad 3(2+7). \qquad 4. \quad 3(a+6). \qquad 7. \quad x(3+7+10).$$

$$2. \quad 5(3+4+5). \qquad 5. \quad 11(h+k). \qquad 8. \quad 15(x+y+z).$$

$$3. \quad 8(5+9+7). \qquad 6. \quad 4(a+b+c). \qquad 9. \quad 20(m+n+p).$$

16. The difference of two numbers in Arabic figures may likewise be multiplied by a given number in either of two ways.

$$\text{E.g.} \quad 6(8-3) = 6 \cdot 5 = 30,$$

$$\text{or} \quad 6(8-3) = (6 \cdot 8) - (6 \cdot 3) = 48 - 18 = 30.$$

In the case of numbers represented by letters evidently the second process only is available.

$$\text{E.g.} \quad 6(r-t) = 6r - 6t \text{ and } a(c-d) = ac - ad.$$

Perform as many as possible of the following multiplications in two ways:

$$1. \quad 7(9-2). \qquad 4. \quad 17(18-11). \qquad 7. \quad 5(x-1). \qquad 10. \quad m(r-s).$$

$$2. \quad 12(17-7). \qquad 5. \quad 9(a-2). \qquad 8. \quad 3(y-2). \qquad 11. \quad x(y-z).$$

$$3. \quad 5(12-8). \qquad 6. \quad 8(h-4). \qquad 9. \quad a(c-d). \qquad 12. \quad t(u-v).$$

The foregoing examples illustrate

Principle II

17. Rule. *To multiply the sum or difference of two numbers by a given number, multiply each of the numbers separately by the given number, and add or subtract the products.*

EXERCISES

1. Multiply $5 + 7 + 11$ by 3 without first adding, and then check by performing the addition before multiplying.

2. Multiply $m + n$ by 4 and check for $m = 5$, $n = 7$.

$$4(m + n) = 4m + 4n$$

Check.

$$4(5 + 7) = 4 \cdot 12 = 48, \text{ also}$$

$$4(5 + 7) = (4 \cdot 5) + (4 \cdot 7) = 20 + 28 = 48.$$

✓ 3. Multiply $x + y$ by r and check for $x = 2$, $y = 4$, $r = 6$.

4. Multiply $r + s$ by k and check for $r = 4$, $s = 5$, $k = 6$.

5. Multiply $a + b + c$ by m and check for $a = 3$, $b = 2$, $c = 1$, $m = 4$.

6. Multiply $m - n + 2$ by c and check for $m = 5$, $n = 2$ and $c = 3$.

7. Multiply $a - b - c$ by d and check for $a = 10$, $b = 3$, $c = 4$ and $d = 8$.

8. Multiply $r + s + 9 - t$ by c and check for $r = 1$, $s = 8$, $t = 3$, and $c = 2$.

Find the following products:

9. $8(13 - 5)$.

16. $5(7 + a - b)$.

10. $2(12 + 41 - 36)$.

17. $3(x + 2 - y)$.

11. $9(a + 8 - b)$.

18. $8(a + b - c + d)$.

12. $3(a + x + y - 1)$.

19. $8(9 - a + b + 3)$.

13. $3(a + b - c)$.

20. $a(3 + b - c)$.

14. $a(18 - 7)$.

21. $c(m - n + p)$.

✓ 15. $7(3 + 8 + 9 - a)$.

22. $19(8 - x - y + z)$.

DIVISION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

18. In dividing the sum or difference of two numbers by a given number, when these are represented by Arabic figures, the process may be carried out in two ways. Thus,

$$(1) \quad (12 + 8) \div 2 = 20 \div 2 = 10,$$

or $(12 + 8) \div 2 = (12 \div 2) + (8 \div 2) = 6 + 4 = 10.$

$$(2) \quad (20 - 12) \div 4 = 8 \div 4 = 2,$$

or $(20 - 12) \div 4 = (20 \div 4) - (12 \div 4) = 5 - 3 = 2.$

Describe each of these two ways of dividing a sum or difference by a number. How do the results compare?

If the numbers in the dividend are represented by letters, the division can usually be carried out only in the second manner shown above.

E.g. $(r + t) \div 5 = (r \div 5) + (t \div 5)$, or, $\frac{r+t}{5} = \frac{r}{5} + \frac{t}{5}.$

In either case this is read: *r plus t divided by 5 equals r divided by 5 plus t divided by 5.*

In this manner perform each of the following divisions in two ways when possible and check the results:

1. $(16 + 12) \div 4.$ 3. $(x - y) \div 6.$ 5. $(m - r) \div a.$

2. $(20 - 10) \div 5.$ 4. $(x + z) \div 3.$ 6. $(m + r) \div b.$

These examples illustrate

Principle III

19. Rule. *To divide the sum or difference of two numbers by a given number divide each number separately and add or subtract the quotients.*

EXERCISES

1. Divide $72 + 56$ by 8 without first adding.
2. Divide $144 - 36$ by 12 without first subtracting.
3. Divide $r + t$ by 5 and check the quotient when $r = 15$, $t = 25$; also when $r = 60$, $t = 75$.

4. Multiply $7 + 9$ by 3 without first adding 7 and 9.
5. Multiply $25 - 8$ by 5 without first subtracting.
6. Find the product of 12 and $a + b$, checking the result when $a = 5$, $b = 7$.

Perform the following indicated operations:

7. $3(a + b + c + d)$. Check for $a = 1$, $b = 2$, $c = 3$, $d = 4$.
8. $7(r - s + t - x)$. Check for $r = t = 5$, $s = x = 4$.
9. $(m + n + r) \div 4$. Check for $m = 64$, $n = 32$, $r = 8$.
10. $(x + y + z) \div 5$. Check for $x = 100$, $y = 50$, and $z = 25$.
11. $74rs - 67rs - 2rs - 3rs$. 13. $a(4 - d + b + c + 3)$.
12. $(63 - 35 - 14 + 21) \div 7$. 14. $(21 - x - y + 3 + c)k$.
15. $49pq + 18pq - 62pq + 3pq$.
16. $13xyz + 3xyz - 8xyz - 7xyz$.
17. $(q + r + s + t - a - b) \div c$.
18. $35lm - 33lm + 7lm - 3lm - 2lm$.
19. $k(l + m + n + r - s - t)$.
20. $a(c + d - c + f - g)$.
21. $27abc - 19abc - 4abc + 8abc$.
22. $(a + r + s - t - q) \div 3$.
23. $(12 - x + y - z) \div 6$.
24. For what values of a , b , c , d are the following equalities true?

$$(a) \quad ab + ac + ad = a(b + c + d).$$

$$(b) \quad ab + ac - ad = a(b + c - d).$$

$$(c) \quad \frac{a + b + c}{d} = \frac{a}{d} + \frac{b}{d} + \frac{c}{d}.$$

Historical Note. The fundamental character of Principles II and III was not fully appreciated until the first part of the last century. Principle II states what is called the **Distributive Law of Multiplication** with respect to addition and subtraction. That is, the multiplier is *distributed* over the multiplicand. The name was first used by a Frenchman F. J. Servois, in a paper published in 1814. Principle III states the same law for division. Compare notes on pages 57, 67.

MULTIPLICATION OF A PRODUCT

20. In arithmetic to multiply a product like $3 \cdot 5$ by 2, we first multiply 3 by 5 and this result by 2.

Thus, $2 \cdot (3 \cdot 5) = 2 \cdot 15 = 30.$

We cannot, however, multiply $3n$ by 2 in this way. But it is obvious that twice $3n$ is $3n + 3n = 6n.$

Thus, if $n = 5,$ $2 \cdot (3 \cdot 5) = 6 \cdot 5.$

Hence in multiplying the product $3 \cdot 5$ by 2 we may multiply the factor 3 *only*. Would you get the same result if you multiplied the factor 5 only?

Example. Find the product $2(3 \cdot 4 \cdot 5)$ in as many ways as possible.

Solution:

$$2(3 \cdot 4 \cdot 5) = 2 \cdot 60 = 120, \text{ and } 2(3 \cdot 4 \cdot 5) = 3 \cdot 8 \cdot 5 = 120,$$

$$\text{Also } 2(3 \cdot 4 \cdot 5) = 6 \cdot 4 \cdot 5 = 120, \text{ and } 2(3 \cdot 4 \cdot 5) = 3 \cdot 4 \cdot 10 = 120.$$

In like manner find the following products:

1. $5(3 \cdot 7).$ 2. $8(2 \cdot 3 \cdot 4).$ 3. $9(2 \cdot 5).$ 4. $6(5 \cdot 3).$
5. $3(5 \cdot 2).$ 6. $4(7 \cdot 2 \cdot 3).$ 7. $7(2 \cdot 4 \cdot 3).$ 8. $5(2 \cdot 5 \cdot 3).$

These examples illustrate

Principle IV

21. **Rule.** *To multiply the product of several factors by a given number, multiply any one of the factors by that number.*

22. Principles IV and II should be carefully contrasted, as in the following example:

$$2(2 \cdot 3 \cdot 5) = 4 \cdot 3 \cdot 5 = 2 \cdot 6 \cdot 5 = 2 \cdot 3 \cdot 10,$$

but

$$2(2 + 3 + 5) = 4 + 6 + 10.$$

In multiplying the product of several numbers we operate upon any one of them, but in multiplying the sum or difference of numbers we operate upon each of them.

EXERCISES

X Multiply as many as possible of the following in two or more ways. Check where letters are involved.

- | | | |
|-----------------------------|-----------------------|-------------------|
| 1. $7(3 \cdot 4 \cdot 5)$. | 5. $3(5xy)$. | 9. $15(7ab)$. |
| 2. $8(7 \cdot 2 \cdot 3)$. | 6. $3(8xyz)$. | 10. $3(4mn)$. |
| 3. $9(2 \cdot 3 \cdot 4)$. | 7. $4(19 \cdot 25)$. | 11. $7(2xy)$. |
| 4. $5(2ab)$. | 8. $5(7abc)$. | 12. $2(16rs)$. + |

DIVISION OF A PRODUCT

23. Division of the product of several factors by a given number may be performed in various ways:

E.g. $(4 \cdot 6 \cdot 10) \div 2 = 240 \div 2 = 120$.

Also $(4 \cdot 6 \cdot 10) \div 2 = 2 \cdot 6 \cdot 10 = 120$,

$(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 3 \cdot 10 = 120$,

and $(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 6 \cdot 5 = 120$.

Note that in each case only one factor is divided.

Perform each of the following divisions in more than one way where possible:

- | | | |
|-----------------------------------|--------------------------------------|--------------------------------------|
| 1. $(5 \cdot 8 \cdot 3) \div 2$. | 4. $(11 \cdot 20 \cdot 16) \div 4$. | 7. $(10 \cdot 35 \cdot 3) \div 5$. |
| 2. $20abc \div 4$. | 5. $14xyz \div 7$. | 8. $14xyz \div x$. |
| 3. $12abc \div 3$. | 6. $12abc \div c$. | 9. $(12 \cdot 40 \cdot 13) \div 8$. |

These examples illustrate

Principle V

24. **Rule.** *To divide the product of several factors by a given number divide any one of the factors by that number.*

25. Principle V is already known in arithmetic in the process called cancellation.

Thus, in the fraction $\frac{2 \cdot 6 \cdot 9}{3}$, 3 may be canceled out of either 6 or 9, giving $\frac{2 \cdot 6 \cdot 9}{3} = 2 \cdot 2 \cdot 9$ or $2 \cdot 6 \cdot 3$.

EXERCISES

1. Contrast Principles III and V.

By Principle V, $(4 \cdot 6 \cdot 8) \div 2 = 2 \cdot 6 \cdot 8 = 4 \cdot 3 \cdot 8 = 4 \cdot 6 \cdot 4$.

By Principle III, $(4 + 6 + 8) \div 2 = 2 + 3 + 4$.

That is *in dividing the product of several numbers by another number we operate upon any one of them, but in dividing their sum or difference we operate upon each of them.*

x Perform the following indicated operations:

2. $5abc + a$.
3. $15xy + 3$.
4. $5(2x + 3y)$.
5. $(33x - 44y) + 11$.
6. $(78s - 39t) + 13$.
7. $8(3a + 2y)$.
8. $15(a - b + c) + 5$.
9. $3a(7 - c + b) + a$.
- x 10. $5bc(d - e + 3) \div bc$.
11. $19(3a - 6b + 9c) + 3$.
12. $13(8 - 4b + 12a) + 4$.
13. $14(7 - 7m + 14n) + 7$.
14. $12a(3b - 3c + 9) + 3$.
15. $24(16b - 8c + 24d) + 8$.
16. Divide $7a \cdot 14b \cdot 21c$ by 7 in three different ways.
17. Add $5a$, $\frac{12a}{2}$, $\frac{21a}{3}$, and $\frac{18a}{6}$, using Principles V and I.
18. From $\frac{28xy}{4}$ subtract $\frac{21xy}{7}$, using Principles V and I.
19. From $\frac{14a}{2} + \frac{10a}{5}$ subtract $\frac{6a}{3}$.
20. Find the sum of $\frac{16x}{8}$, $\frac{20x}{5}$, $\frac{16x}{4}$, $7x$, and $3x$.
21. Find the sum of $\frac{100rs}{10}$, $\frac{90rs}{9}$, and $\frac{25rs}{5}$.
22. From $25xy$ subtract $\frac{13xyz}{z}$.
23. Add $\frac{8abc}{c} + \frac{18ab}{3} + \frac{7abd}{d} + \frac{abe}{e}$.
24. Add $\frac{18a}{6} + \frac{5ba}{b} + \frac{4xa}{x} + \frac{am}{m}$.

ORDER OF INDICATED OPERATIONS

26. In a succession of indicated operations the final result depends in some cases upon the *order* in which the steps are taken :

Thus, $6 + 3 \cdot 8$ would give $9 \cdot 8 = 72$, if the addition were performed first; and would give $6 + 24 = 30$, if the multiplication were performed first.

Similarly, $24 \div 2 \cdot 3$ would give $12 \cdot 3 = 36$, if the division were performed first; and would give $24 \div 6 = 4$, if the multiplication were performed first.

27. However, *when only additions and subtractions are involved and no symbols of aggregation occur, the result is the same no matter in what order the operations are performed.*

Test this statement by performing each of the following indicated operations in several different orders :

1. $8 - 3 + 14 - 4$.

3. $7 - 6 + 1 + 3$.

2. $13 + 4 - 3 + 2$.

4. $15 - 12 + 4 - 2$.

A similar statement holds *in some cases* when only multiplications and divisions are involved.

Thus, $12 \times 6 \div 3 = 24$ no matter in which order the operations are performed, while $24 \div 2 \cdot 3$ may equal 4 or 36.

In case of doubt symbols of aggregation should be used to show the order intended.

Thus, $24 \div (2 \cdot 3) = 4$, while $(24 \div 2) \cdot 3 = 36$.

And $18 \div (6 \div 2) = 6$, while $(18 \div 6) \div 2 = \frac{3}{2}$.

The following rule is in accordance with universal custom :

28. *In an expression involving additions, subtractions, multiplications, and divisions without any symbols of aggregation, all multiplications and divisions are to be performed before any additions or subtractions.*

E.g. $5 + 3 \cdot 4 - 8 \div 2 = 5 + 12 - 4 = 13$, and $8 - 2 \div 2 - 2 \times 2 + 3 = 8 - 1 - 4 + 3 = 6$.

EXERCISES

× Perform the following indicated operations:

1. $20 + 5 - 3 + 4 - 2$.

5. $(28 + 4) \cdot 2 + 10$.

2. $12 + 3 - 2 + 5$.

6. $5 \cdot 3 + 2 \cdot 7 - 19$.

3. $(28 \cdot 4) + 2 + 10$.

7. $26 + 2 - 6 \cdot 2$.

4. $28(4 + 2) + 10$.

8. $18 - 2 \cdot 6 - 8 + 4 + 20$.

DRILL EXERCISES

NOTE.—The pages headed “Drill Exercises” are intended to afford general practice in performing algebraic operations. Each such page consists of miscellaneous exercises on the various topics that precede.

If $a = 5$, $b = 3$, $c = 2$, find the numerical value of each of the following:

1. $ac + bc$ and $(a + b)c$.

5. $a(b - c)$ and $ab - ac$.

2. $ac - bc$ and $(a - b)c$.

6. $\frac{ab}{c}$, $\frac{a}{c} \times b$ and $a \times \frac{b}{c}$.

3. abc , $a \times (bc)$ and $b \times (ac)$.

7. $\frac{a + b}{c}$ and $\frac{a}{c} + \frac{b}{c}$.

4. $a(b + c)$ and $ab + ac$.

8. $\frac{a - b}{c}$ and $\frac{a}{c} - \frac{b}{c}$.

Perform the following indicated operations and state what principle is used in each case:

9. $3ax + 7ax - 4ax$.

15. $(24x + 9y) + 3$.

10. $7by + 4by - 9by$.

16. $(6a - 4b) + 2$.

11. $3y - 2y + y$.

17. $(3xy + 2bx) + x$.

12. $c(4a + 2b)$.

18. $(7abc + 2abd) + ab$.

13. $5(16a - 3b)$.

19. $2a(7x + 3y - 4)$.

14. $a(11 - b)$.

20. $3x(2a - 5b + c)$.

What principles are used in the following operations?

21. $3 \cdot 5ab = 15ab$.

23. $3t + 15t = 18t$.

22. $16xy + x = 16y$.

24. $78h - 41h = 37h$.

29. Importance of the Principles. The five principles studied in this chapter, together with others which will be introduced when needed, will be found of increasing importance as we proceed. Your success in the further study of algebra will depend in no small degree upon the clearness with which you understand their real significance. The most effective way to master them is by means of simple numerical examples such as were used in introducing each one. Make a list of these principles in abbreviated form for yourself and note how frequently your own errors and those of your classmates are due to direct violations of one or more of them.

REVIEW QUESTIONS

1. How would $3 \cdot 5$ and $7 \cdot 5$ be added in arithmetic? Why cannot $3n$ and $7n$ be added in the same manner? State in full the principle by which $3n$ and $7n$ are added. In this example what number is represented by n ? Test the identity $3n + 7n = 10n$ by substituting any convenient value for n .

2. What kind of numbers may be added by Principle I? Have the numbers ac and bc a common factor? What is it? What is the coefficient of this common factor in each? What is the sum of these coefficients? Is the equality $ac + bc = (a + b)c$ true no matter what numbers are represented by a , b , and c ? When this can be said of an equality, what is it called?

3. How is $5 \cdot 9$ subtracted from $11 \cdot 9$ in arithmetic? In what different manner may this operation be performed? Why is it sometimes necessary to perform subtraction in the second way? In the identity $31x - 12x = 19x$, what number is represented by x ? Test the equality by substituting any convenient number for x . Is this equality true for every value of x ?

Principle I may be conveniently abbreviated as follows:

$$\begin{cases} ac + bc = (a + b)c, \\ ac - bc = (a - b)c. \end{cases}$$

4. How is $11 + 3$ multiplied by 4 in arithmetic? In what different way may this operation be performed? Why is it sometimes necessary to multiply in the second way? State in full the principle by which $a + 8$ is multiplied by 7.

Principle II may be abbreviated thus:

$$\begin{cases} c(a + b) = ca + cb, \\ c(a - b) = ca - cb. \end{cases}$$

Notice that the identities in Principle II are the same as those in I read in reverse order.

5. How is $12 + 18$ divided by 6 in arithmetic? In what different way may this division be performed? Why is it sometimes necessary to perform division in the second way? State in full the principle used in performing the operation $(6x + 9y) \div 3$. How do you divide $(6x - 9y)$ by 3?

Principle III may be abbreviated thus:

$$\left\{ \frac{a + b}{k} = \frac{a}{k} + \frac{b}{k}, \quad \frac{a - b}{k} = \frac{a}{k} - \frac{b}{k} \right\}.$$

6. How is the product $2 \cdot 3 \cdot 5$ multiplied by 4 in arithmetic? In what different way may this multiplication be performed? Why is it ever performed in the second way? What are the factors of the number ab ? How is the product of two numbers multiplied by another number? Should *both* factors be multiplied by the number or only one? Is it permissible to multiply *either one* we choose? Principle IV is abbreviated thus:

$$k \times (ab) = (ka) \times b = a \times (kb).$$

7. Divide $2 \cdot 4 \cdot 6 \cdot 20$ by 2 without first performing the multiplication indicated in $2 \cdot 4 \cdot 6 \cdot 20$. Do this in several ways and show that all the quotients obtained are equal. State in full the principle used.

Principle V is abbreviated thus:

$$(ab) \div k = \frac{a}{k} \times b = a \times \frac{b}{k}.$$

8. Contrast Principles II and IV; also III and V.

Historical Note. The oldest mathematical manuscript in existence is a papyrus now in the British Museum which was written by Ahmes, an Egyptian priest, possibly as early as 3000 B.C. This contains, besides work in arithmetic, some very simple problems involving the beginnings of algebra. We have no knowledge that the early Egyptians advanced beyond these simplest rudiments.

Singularly enough, the Greeks failed to make many important contributions to algebra, though they greatly advanced the sister science geometry. Diophantus, who lived in Alexandria about 320 A.D., is the only Greek whose contribution to algebra is worthy of note. Much of his work, however, is not of a kind which can be included in an elementary book such as this.

These beginnings in Egypt and Greece were so insignificant that the Hindus may be said to have created algebra. Most of the material contained in beginners' books such as the present one was known to the Hindus by the year 600 A.D. The Arabs learned algebra from the Hindus and brought it to Europe in the eighth century. It was not until the twelfth century, however, that the Arabian books on algebra were translated into Latin and thus became accessible to Europeans.

The Arabs used very few symbols. Number expressions were written out fully in words. Thus, instead of $x^2 + 10x = 39$ they wrote *A square and ten of its roots are equal to thirty-nine*. In this respect they took a step backward, for the Hindus, from whom the Arabs learned much of their algebra, had made considerable use of symbols. As we have seen (page 2) it was not until the middle of the seventeenth century that the present symbols came into general use.

Letters were used now and then from the earliest time as abbreviations for words representing numbers. But the systematic use of letters for this purpose began with the great French algebraist, François Vieta (1540-1607).

It must not be supposed, however, that the use of these symbols was adopted suddenly. It is a curious fact of history that people cling to the old, even though the new is vastly superior. It was only after René Descartes (1596-1650) had used these symbols in his works that they came to be used in texts on algebra (see also page 50).

CHAPTER II

EQUATIONS AND PROBLEMS

The principles developed in the last chapter will now be used in the solution of equations and problems.

SOLUTION OF EQUATIONS

30. Definitions. Equalities in which letters are used as number symbols are of two kinds, namely:

(1) Identities, such as $3(5x + 6y) = 15x + 18y$, which holds for all values which may be assigned to x and y . See § 13.

(2) Equalities, such as $3x = 18$, which holds if, and only if, $x = 6$.

The equality $3x = 18$ is said to be **satisfied** by $x = 6$, because this value of x reduces both members to the same number, 18.

Ex. 1. Is $x + 4 = 9$ satisfied by $x = 4$? by $x = 5$? by $x = 6$?

Ex. 2. Is $7x + 9 = 3x + 25$ satisfied by $x = 2$? by $x = 3$? by $x = 4$?

Ex. 3. Is $\frac{x}{4} + 3 = 6$ satisfied by $x = 4$? by $x = 8$? by $x = 12$?

Ex. 4. Is $\frac{x+1}{3} + 2x - 1 = 2x$ satisfied by $x = 2$? by $x = 4$? by $x = 1$?

31. An equality which is satisfied only when certain particular values are given to one or more of its letters is called a **conditional equality** with respect to those letters.

E.g. $3x + 5 = 35$ is an equality only on the condition that $x = 10$. $x + y = 10$ is an equality for certain pairs of values of x and y like 1 and 9, 2 and 8, 3 and 7, 5 and 5, but certainly not for all values of x and y ; for instance, not for $x = 3$ and $y = 8$.

32. A conditional equality is called an **equation**, and a letter whose particular value is sought to satisfy an equation is called an **unknown** in that equation.

The equations at present considered contain only one unknown.

33. To solve an equation in one unknown is to find the value or values of the unknown which satisfy it. Such a value of the unknown is called a **root** or **solution** of the equation.

The following examples illustrate methods used in solving equations.

Ex. 1. Solve the equation $x - 5 = 9$. (1)

Solution. This equation states that 9 is 5 less than the number x , that is, if 9 be increased by 5, the result is x .

Hence, $x = 14$ is the solution of equation (1).

This result may be obtained by adding 5 to each member of equation (1),
thus

$$x + 5 - 5 = 9 + 5,$$

or $x = 14$. (2)

Check. Substitute $x = 14$ in equation (1) and get

$$14 - 5 = 9.$$

Hence the equation is satisfied by $x = 14$.

Ex. 2. Solve the equation $x + 7 = 12$. (1)

Solution. This equation states that 12 is 7 more than the number x , that is, if 12 be diminished by 7, the result is x .

Hence, $x = 5$ is the solution of equation (1).

This result may be obtained by subtracting 7 from both members of (1),
thus

$$x + 7 - 7 = 12 - 7,$$

or $x = 5$. (2)

Check. Substitute $x = 5$ in (1) and get

$$5 + 7 = 12.$$

Hence the equation is satisfied by $x = 5$.

Ex. 3. Solve the equation $\frac{1}{3}x = 7$. (1)

Solution. This equation states that one-third of the number x is 7, that is, x is three times 7, or 21.

Hence $x = 21$ is the solution of equation (1).

This result may be obtained by multiplying both members of (1) by 3,
thus

$$3 \cdot \frac{1}{3}x = 3 \cdot 7,$$

or $x = 21$. (2)

Check. Substitute $x = 21$ in (1) and get

$$\frac{1}{3} \cdot 21 = 7.$$

Hence the equation is satisfied by $x = 21$.

Ex. 4. Solve the equation $5x = 30$. (1)

Solution. This equation states that 5 times the number x is 30, that is, x is one-fifth of 30, or 6.

Hence, $x = 6$ is the solution of the equation.

This result may be obtained by dividing both members of equation (1) by 5, thus

$$\frac{5x}{5} = \frac{30}{5},$$

or $x = 6$. (2)

Check. Substitute $x = 6$ in equation (1) and get

$$5 \cdot 6 = 30.$$

Hence the equation is satisfied by $x = 6$.

34. The above examples illustrate four ways of operating upon both members of an equation, so as to produce in each case a new equation *satisfied by the same value of the unknown as the original equation.*

Each of these operations *changes the value of both members, but changes them both alike.*

In the next example, equations (2) and (3) show two ways of *changing the form, but not the value, of one member alone*, and thus producing a new equation satisfied by the same value of the unknown as the original equation.

Ex. 5. Solve the equation

$$w + 2(w + 5) = 58. \quad (1)$$

By Principle II, $w + 2w + 10 = 58. \quad (2)$

By Principle I, $3w + 10 = 58. \quad (3)$

Subtracting 10 from both members, $3w = 48. \quad (4)$

Dividing both members by 3, $w = 16. \quad (5)$

Check. Putting $w = 16$ in (1), $16 + 2(16 + 5) = 16 + 2 \cdot 21 = 58.$

By substitution verify that $w = 16$ also satisfies equations (2), (3), and (4).

The operations involved in passing from (1) to (2) and (2) to (3) are called **form changes** which leave the value of the left member unaltered.

All the operations involved in Principles I to V are *form changes* of this character. See the list at the end of Chapter I. There are other form changes which will be considered as need arises.

35. The members of an equation may be likened to the **scale-pans of a common balance** in which are placed objects of uniform weight, say tenpenny nails. The scales balance only when the weights are the same in both pans; that is, when the number of nails is the same.

If now the scales are in balance, they will remain so under two kinds of changes in the weights:

(a) When the number of nails in the two pans is increased or diminished by the same amount; corresponding to **like changes** in value of both members of an equation.

(b) When the number in each pan is left unaltered but the nails are rearranged in groups or piles in any manner; corresponding to **form changes** on either member of an equation.

The equation, then, is like a balance, and its members are to be operated upon only in such ways as to preserve the balance.

EXERCISES

Solve the following equations and explain each step involved, as in the above illustrative examples.

Y

- | | | |
|-------------------------|--------------------|--------------------------|
| 1. $7x = 42.$ | 7. $x - 3 = 7.$ | 13. $x + 3x + 4x = 16.$ |
| 2. $11n = 77.$ | 8. $2n + 5 = 15.$ | 14. $5n - 2n + 3n = 40.$ |
| 3. $\frac{1}{2}y = 8.$ | 9. $3y + 8 = 17.$ | 15. $6y + 8y - 3y = 33.$ |
| 4. $\frac{1}{3}x = 18.$ | 10. $x - 4 = 12.$ | 16. $5x + 7 + 2x = 14.$ |
| 5. $\frac{1}{4}n = 20.$ | 11. $2y - 3 = 15.$ | 17. $6y + 8y - 5 = 23.$ |
| 6. $6w = 28.$ | 12. $4w - 5 = 21.$ | 18. $3w - 2w - 2 = 4.$ |

The foregoing examples illustrate

Principle VI

36. Rule. *An equation may be changed into another equation such that any value of the unknown which satisfies one also satisfies the other, by means of any of the following operations:*

- (1) *Adding the same number to both members;*
- (2) *Subtracting the same number from both members;*
- (3) *Multiplying both members by any number not zero;*
- (4) *Dividing both members by any number not zero;*
- (5) *Changing the form of either member in any way which leaves its value unaltered.*

The operations under Principle VI are hereafter referred to in detail by means of the initial letters, *A* for *addition*, *S* for *subtraction*, *M* for *multiplication*, *D* for *division*, and *F* for *form changes* which leave the value of a member unaltered.

Historical Note. The first Arabic algebra known to us dates from the first half of the ninth century. It bore the name *Al-gebr walmukâbâla*. The word *al-gebr*, from which the word *algebra* is derived, means "restoration" and refers to the fact that the same number may be added to or subtracted from each member of an equation. The word "*walmukâbâla*" means the process of simplification or *form changes*, such as have just been described. Thus it appears that the Arabs regarded the solution of equations as the main business of algebra.

DIRECTIONS FOR WRITTEN WORK

37. In solving an equation the successive steps should be written as in the following:

$$\text{Ex. 1. } 25(n+1) + 6(4n-3) = 50 + 31n + 2(3-n) - 9. \quad (1)$$

By F , using Principle II, we obtain from (1)

$$25n + 25 + 24n - 18 = 50 + 31n + 6 - 2n - 9. \quad (2)$$

By F , using Principle I, we obtain from (2)

$$49n + 7 = 29n + 47. \quad (3)$$

Subtracting 7 and $29n$ from each member of (3) and using Principle I, we have

$$20n = 40. \quad (4)$$

Dividing each member of (4) by 20,

$$n = 2. \quad (5)$$

Check. Substitute $n = 2$ in equation (1).

For convenience this work can be abbreviated as follows:

$$25(n+1) + 6(4n-3) = 50 + 31n + 2(3-n) - 9. \quad (1)$$

$$\text{By } F, II, \quad 25n + 25 + 24n - 18 = 50 + 31n + 6 - 2n - 9. \quad (2)$$

$$\text{By } F, I, \quad 49n + 7 = 29n + 47. \quad (3)$$

$$\text{By } S | 7, 29n, \quad 20n = 40. \quad (4)$$

$$\text{By } D | 20, \quad n = 2. \quad (5)$$

$S | 7, 29n$ means that 7 and $29n$ are each to be subtracted from both members of the preceding equation. $D | 20$ means that the members of the preceding equation are to be divided by 20.

Similarly, in case we wish to indicate that 6 is to be added to each member of an equation, we would write $A | 6$, and if each member is to be multiplied by 8, we would write $M | 8$. It is important that the nature of each step be recorded in some such manner.

$$\text{Ex. 2. } 17n + 4(2+n) - 6 = 5(4+n) - 5 + 3n. \quad (1)$$

$$\text{By } F, II, \quad 17n + 8 + 4n - 6 = 20 + 5n - 5 + 3n. \quad (2)$$

$$\text{By } F, I, \quad 21n + 2 = 15 + 8n. \quad (3)$$

$$\text{By } S | 2, 8n, \quad 21n - 8n = 15 - 2. \quad (4)$$

$$\text{By } F, I, \quad 13n = 13. \quad (5)$$

$$\text{By } D | 13, \quad n = 1. \quad (6)$$

Check. Substitute $n = 1$ in equation (1).

38. By use of Principle VI, a term may be **transposed** from one member of an equation to the other by changing its sign.

E.g. in deriving equation (4) from (3) in the last solution, $8n$ was subtracted from both sides by mentally *dropping* it on the right and *indicating* its subtraction on the left. Likewise when 2 is subtracted from both sides it *disappears* on the left and *appears* on the right with the *opposite* sign. Each of these indicated subtractions might have been performed *mentally*, thus writing equation (5) directly from (3).

After a little practice this shorter process of *transposing terms* and *combining similar terms mentally* should always be used.

39. An equation may be translated into a problem. For example, the equation $21x + 2 = 8x + 15$ may be interpreted as follows: *Find a number such that 21 times the number plus 2 is 15 greater than 8 times the number.*

EXERCISES

Solve the following equations, putting the work in a form similar to the above and checking each result. Translate the first five into problems.

1. $13x + 40 - x = 88.$
2. $3x + 9 + 2x + 6 = 18 + 4x.$
3. $5x + 3 - x = x + 18.$
4. $13y + 12 + 5y = 32 + 8y.$
5. $4m + 6m + 4 = 9m + 6.$
6. $7m + 18 + 3m = 12 + 2m + 38.$
7. $3y + 4 + 2y + 6 = y + 7 + y + 3 + 30.$
8. $5x + 3 + 2x + 3 = 2x + 5 + 3x + 3 + x.$
9. $2x + 4x + 9 - x + 6 = 20 + 2x + 5 + x.$
10. $18 + 6m + 30 + 4m = 4m + 8 + 12 + 3m + 3 - m + 29$
11. $y + 72 + 45y = 106 + 12y.$
12. $42x + 56 = 20x + 122.$
13. $6x + 8 + x + 4 + 5x = 7x + 32 - x - 20.$

14. $32x + 4 + 7x = 66 + 3x + 5x$.
15. $12m + 3 - 3m = 38 + 2m$.
16. $15m + 3 - 2m + 7 = 3m + 60$.
17. $a + 7 + 3a = 2a + 45$.
18. $5b + 30 + 6b = 3b + 150$.
19. $3c + 18 + 14c = 6c + 51$.
20. $17x + 4 + 3x = 7x + 30$.
21. $7(m + 6) + 10m = 42 + 5m + 24$.
22. $6x + 4(4x + 2) + 3(2x + 7) = 85$.
23. $8 + 7(6 + 6n) + 2n = 2(4n + 5) + 18n + 49$.
24. $5(9x + 3) + 4(3x + 2) = 18x + 36$.
25. $7(3x + 2) + 13 = 5(2 - x) + 43$.
26. $15 + 3(3 + x) + 2(2 + 6x) = 3(3x + 4) + 28$.
27. $11(x + 5) + 3(3x - 1) = 7(x + 2) + 4x + 47$.

SOLUTION OF PROBLEMS

40. One great object in the study of algebra is to **simplify the solution of problems**. This is done by using letters to represent the unknown numbers, by stating the problem in the form of an equation, and by arranging the successive steps of the solution in an orderly manner.

41. Illustrative Problem. 1. The shortest railway route from Chicago to New York is 912 miles. How long does it take a train averaging 38 miles an hour to make the journey?

Solution. Let t be the number of hours required. Then $38t$ is the distance traveled in t hours. But 912 miles is the given distance traveled.

$$\text{Hence} \qquad 38t = 912. \qquad (1)$$

$$\text{By } D \mid 38 \qquad t = 24. \qquad (2)$$

Check. Substitute $t = 24$ in equation (1) and get

$$38 \cdot 24 = 912.$$

Hence $t = 24$ satisfies the conditions of the problem.

42. Illustrative Problem. 2. For how many years must \$ 850 be invested at 5 % simple interest in order to yield \$ 255 ?

Solution. From arithmetic, we have

$$\text{principal} \times \text{rate} \times \text{time} = \text{interest},$$

or $p r t = i$. See page 3, Ex. 7.

Hence, from the conditions of this problem,

$$850 \times .05 \times t = 255, \quad (1)$$

or $42.5 t = 255$.

$$\text{By } D | 42.5 \quad t = 6. \quad (2)$$

Check. Substitute $t = 6$ in (1) and find

$$42.5 \times 6 = 255.$$

Hence 6 years satisfies the conditions of the problem.

43. Illustrative Problem. 3. A boy, an apprentice, and a master workman have the understanding that the apprentice shall receive twice as much as the boy and the master workman five times as much as the boy. How much does each get, if the total amount received for a piece of work is \$104.

Solution. Let n represent the number of dollars received by the boy.

Then, $2n$ is the number of dollars received by the apprentice,
and $5n$ is the number of dollars received by the master workman.

Hence, $n + 2n + 5n$ and 104 are number expressions, each representing the total amount received.

$$\text{Therefore,} \quad n + 2n + 5n = 104. \quad (1)$$

$$\text{By Principle I,} \quad 8n = 104. \quad (2)$$

$$\text{By } D | 8, \quad n = 13. \quad (3)$$

Hence, the amount received by the boy is 13 dollars, by the apprentice 26 dollars, and by the master workman 65 dollars.

Check. By the conditions of the problem the sum of the amounts obtained should be \$104; the apprentice should receive twice as much as the boy and the master workman five times as much as the boy. That is, we should have

$$13 + 26 + 65 = 104, \quad 26 = 2 \cdot 13 \text{ and } 65 = 5 \cdot 13.$$

PROBLEMS

Solve the following problems by means of equations and check each result by verifying that it satisfies the conditions of the problem.

✓ 1. Five times a certain number equals 80. What is the number? Use n for the number.

2. Twelve times a number equals 132. What is the number?

3. A tank holds 750 gallons. How long will it take a pipe discharging 15 gallons per minute to fill the tank?

4. The cost of paving a block on a certain street was \$7 per front foot. How long was the block, if the total cost was \$4620?

5. A city lot sold for \$7500. What was the frontage, if the selling price was \$225 per front foot?

6. An encyclopedia contains 18,000 pages. How many volumes are there, if they average 750 pages to the volume?

x 7. For how many years must \$3500 be invested at 6% simple interest to yield \$2205?

8. At what rate must \$2500 be invested for 3 years in order to yield \$412.50?

Suggestion. By the conditions of the problem $2500 \times r \times 3 = 412.50$.

9. At what rate must 6800 be invested for 7 years in order to yield \$2380?

10. How many dollars must be invested for 5 years at $4\frac{1}{2}\%$ simple interest to yield \$351?

Suggestion. By the conditions of the problem $p \times .04\frac{1}{2} \times 5 = 351$.

11. How many dollars must be invested for 6 years at $4\frac{3}{4}\%$ simple interest to yield \$2422.50?

12. A cut in an embankment is 500 yards long and 4 yards deep. How wide is it if 18,760 cubic yards are removed?

Suggestion. From arithmetic, length \times width \times height = volume or $lwh = v$.

13. How deep is a rectangular cistern which holds 500 cubic feet of water, if it is 6 feet wide and 8 feet long?

14. How long is a box containing 2240 cubic inches, if its width is 14 inches and its depth 10 inches?

15. If n is a number, how do you represent 10 times that number?

16. If n is a number, how do you represent that number plus 3 times itself?

17. If n is a number, how do you represent 5 times that number plus 3 times the number plus 8 times the number?

18. The greater of two numbers is 5 times the less, and their sum is 180. What are the numbers?

19. A number increased by twice itself, 4 times itself, and 6 times itself, becomes 429. What is the number?

20. A father is 3 times as old as his son, and the sum of their ages is 48 years. How old is each?

21. In a company there are 39 persons. The number of children is twice the number of grown people. How many are there of each?

22. How many dollars will amount to \$620 in 4 years at 6% simple interest?

Solution. From arithmetic we have

$$\text{amount} = \text{principal} + \text{interest},$$

or

$$a = p + i = p + prt.$$

Hence, by the conditions of the problem,

$$620 = p + .06 \times 4 \times p.$$

By Principle II, $620 = p(1 + .24).$

By D ,

$$p = \frac{620}{1.24} = 500.$$

23. Find what principal invested for 6 years at $4\frac{1}{2}\%$ simple interest will amount to \$1270.

24. Find what principal invested for 12 years at $5\frac{1}{2}\%$ simple interest will amount to \$4150.

ALGEBRAIC REPRESENTATION OF NUMBERS

Skill in translating problems into equations depends upon attention to the following points:

(1) *Read and understand* clearly the statement of the problem, as it is given in words.

(2) *Select the unknown number*, and represent it by a suitable letter, say the initial letter of a word which will keep its meaning in mind. If there are more unknown numbers than one, try to express the others in terms of the one first selected.

(3) Find two number expressions which, according to the problem, represent the same number, and set them equal to each other, *thus forming an equation*.

Special care is needed in expressing the various numbers involved in a problem in terms of a single unknown number, as illustrated in the following exercises.

1. If n is a number, represent in symbols a number 7 greater than n ; 5 less than n ; 8 times as great as n ; one-third as great as n .

2. Write in symbols n increased by k ; n decreased by k ; n multiplied by k ; n divided by k .

3. If the sum of two numbers is 10 and one of them is x , what is the other number?

4. If two numbers differ by 6 and the smaller is x , what is the other number?

5. If two numbers differ by 6 and the greater is x , what is the other number?

6. If A has m dollars, and B has 15 dollars more than A, how do you represent B's money? If C's money is twice B's, how do you represent C's money?

7. If n is an integer, that is, a whole number, how do you represent the next higher integer? The second higher? The next lower? The second lower?

8. What is the value of $2n$, for $n = 1, 2, 3, 4, 5, 6$, etc.? If n is any integer, the number represented by $2n$ is an *even* integer.

9. If $2n$ is any even integer, represent the next higher even integer.

10. Represent each of four consecutive even integers, the smallest of which is $2n$.

11. What is the value of $2n + 1$, for $n = 1, 2, 3, 4, 5$, etc.? If n is any integer, the number represented by $2n + 1$ is an *odd* integer.

12. If $2n + 1$ is any odd integer, represent the next higher odd integer.

13. Represent four consecutive odd integers, the smallest of which is $2n + 1$.

14. If x is a number, express in terms of x a number 5 less than 3 times x ; also a number 5 times as great as x diminished by 3.

15. The length of a rectangle is 3 feet greater than its width. If w is the width, how do you represent its length?

16. If w and l are the width and length respectively of a rectangle, how do you represent its perimeter? (The perimeter of a rectangle means the sum of the lengths of its four sides.)

17. Express the perimeter of a rectangle in terms of its width w , if its length is 10 inches greater than its width.

18. Express the perimeter of a rectangle in terms of its length l , if the length is 6 inches greater than the width.

PROBLEMS

Check each solution by finding whether the result satisfies the conditions stated in the problem:

1. Four times a certain number plus 3 times the number minus 5 times the number equals 48. What is the number?

2. One number is 4 times another, and their difference is 9. What are the numbers?

4 3. Find a number such that when 4 times the number is subtracted from 12 times the number, the remainder is 496.

4. Thirty-nine times a certain number, plus 19 times the number, minus 56 times the number, plus 22 times the number, equals 12. Find the number.

5. There are three numbers whose sum is 80. The second is 3 times the first, and the third twice the second. What are the numbers?

6. There are three numbers such that the second is 11 times the first and the third is 27 times the first. The difference between the second and the third is 64. Find the numbers.

7. There are three numbers such that the second is 8 times the first and the third is 3 times the second. If the second is subtracted from the third, the remainder is 48. Find the numbers.

8. The number of representatives and senators together in the United States Congress, according to the new apportionment, is 531. The number of representatives is 51 more than 4 times the number of senators. Find the number of each.

9. The area of Illinois is 6750 square miles more than 10 times that of Connecticut. The sum of their areas is 61,640 square miles. Find the area of each state.

10. The sum of the horse powers of the steamships *Olympic* and *Mauretania* is 116 thousand. The *Mauretania* has 22 thousand horse-power less than twice that of the *Olympic*. What is the horse-power of each ship?

11. It is twice as far from Boston to Quebec as from Boston to Albany and 3 times as far from Boston to Jacksonville, Florida, as from Boston to Quebec. How far is it from Boston to each of the other three cities, the sum of the distances being 1818 miles?

⑫ Find three consecutive integers whose sum is 144.

13. Find four consecutive integers such that twice the first plus the last equals 48.

⑭ Find three consecutive even integers whose sum is 54.

15. Find three consecutive even integers such that 3 times the first is 12 greater than the third.

16. Find two consecutive integers such that 3 times the first plus 7 times the second equals 217.

17. Find two consecutive integers such that 7 times the first plus 4 times the second equals 664.

18. Find four consecutive odd integers such that 7 times the first equals 5 times the last.

19. The perimeter of a square is 64 inches. Find the length of a side.

20. A rectangle is 4 inches longer than it is wide. Find its length and width if the perimeter is 40 inches.

21. A rectangle is twice as long as wide. Find its dimensions if the perimeter exceeds the length by 60.

22. The length of a rectangle is $1\frac{1}{2}$ times as great as its width. Find its dimensions if the perimeter exceeds the width by 40 inches.

23. The width of a rectangle is $\frac{3}{4}$ of its length and the perimeter exceeds the length by 50 inches. Find its dimensions.

24. How much must be invested at 6% interest to amount to \$2650 at the end of one year?

25. How much must be invested at 5% simple interest to amount to \$2025 at the end of 7 years?

26. At what rate of interest per year must \$800 be invested to amount to \$1000 in 5 years?

27. Pikes Peak is 3282 feet higher than Mt. Ætna, and Mt. Everest is 708 feet more than twice as high as Pikes Peak. The sum of the altitudes of Mt. Ætna and Mt. Everest is 39,867 feet. Find the altitude of each of the three mountains.

28. The melting point of iron is 450 degrees centigrade higher than 5 times that of tin. Three times the number of degrees at which iron melts plus 7 times the number at which tin melts equals 6410. Find the melting point of each metal.

REVIEW QUESTIONS

1. Define equality ; equation ; identity. State in detail how the equation and the identity differ. Give an example of each.

2. What value of x satisfies the equation $x + 4 = 9$? What value of x will satisfy the equation obtained by adding 7 to each member of this equation? by adding 12? 24?

3. If 4 be added to the first member of the equation $x + 4 = 9$, and 6 to the second member, what value of x will satisfy the equation thus obtained?

4. If the *same* number is added to each member of an equation, is the resulting equation satisfied by the same value of the unknown as the first equation?

5. If *different* numbers are added to the members of an equation, is the resulting equation satisfied by the same value of the unknown as the first equation?

6. If the *same* number is subtracted from each member of an equation, is the resulting equation satisfied by the same value of the unknown as the first equation? Illustrate by an example.

7. If *different* numbers are subtracted from the members of an equation, is the resulting equation satisfied by the same value of the unknown as the first equation? Illustrate by an example.

8. Ask and answer questions, similar to the two preceding, about the effect of multiplying both members of an equation by the *same or different* numbers. Illustrate each by examples.

9. Ask and answer similar questions on the effect of dividing the members of an equation by the same or different numbers. Illustrate each by an example.

10. *The process of solving an equation consists in obtaining from it other equations which are satisfied by the same number as the original equation.*

Hence what operations may be performed in solving an equation?

11. State Principle VI in full.

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

44. Thus far the numbers used have been precisely the same as in arithmetic, though their representation by means of letters and some of the methods used in operating upon them are peculiar to algebra.

We now proceed to the study of a **new kind of number**.

Examples. What is the highest temperature you have ever seen recorded on the thermometer? the lowest?

In answering these questions you not only give certain numbers, but you attach to each a certain **quality**. The temperature is **above** zero or **below** zero, that is, the degrees on the thermometer are measured in **opposite** directions from a starting point which is labeled zero.

45. It has been found exceedingly useful in mathematics to extend the number system of arithmetic so as to make it apply directly to cases like this. The opposite qualities involved are designated by the words **positive** and **negative**.

It is commonly agreed to call *above zero* positive and *below zero* negative. Likewise, distances measured to the **right** from a zero point are called positive, and those to the left, negative. See § 48.

46. **Definitions.** The signs $+$ and $-$ stand respectively for the words *positive* and *negative*, and numbers marked with these signs are called **positive** and **negative numbers** respectively.

Thus, 5° above zero is written $+5^{\circ}$, and 15° below zero is written -15° .

When no sign of quality is written, the positive sign is understood.

E.g. $+5^{\circ}$ is usually written 5° .

Positive and negative numbers are sometimes called **signed numbers**, because each such number consists of a numerical part, together with a **sign of quality** expressed or understood.

The numerical part of a signed number is called its **absolute value**.

Thus, the absolute value of $+3$ and also of -3 is 3.

47. The **integers of arithmetic** may be arranged in a series beginning at zero and extending indefinitely toward the right.

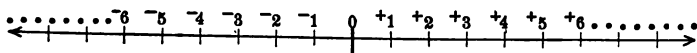
Thus, $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots$

The **integers of algebra** may be arranged in a series beginning at zero and extending indefinitely both to the right and the left.

Thus, $\dots -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \dots$

48. One of the most extensive uses of signed numbers is for marking the points on a straight line. This will also appear later in connection with the graph in Chapter VII.

On an unlimited straight line call some starting point zero, and lay off from this point equal divisions of the line indefinitely both to the right and to the left, as shown in the figure.



In order to describe the position of any one of these division points, we require not only an integer of arithmetic, to specify *how far* the given point is from the point marked zero, but also a **sign of quality** to indicate on which side of this point it is.

E.g. $+7$ marks the division point 7 units to the right of zero, and -5 marks the point 5 units to the left of zero. Such a diagram is called the **scale of signed numbers**.

Fractions would of course be pictured at points between the integral division points, on the right or the left of the scale, according as the fractions are positive or negative.

ADDITION OF SIGNED NUMBERS

49. In arithmetic two numbers are added by starting with one and counting *forward* the number of units in the other.

E.g. To add 3 to 5 we start with 5 and count 6, 7, 8.

In algebra two signed numbers are added in the same manner except that the *direction*, forward or backward, in which we count, is determined by the sign, + or -, of the number which we are adding.

Thus, to add +5 to +7, begin at 7 to the right of the zero point in the scale of signed numbers and count 5 more toward the right, arriving at +12. Thus, $+7 + +5 = +12$, which is read: *Positive 7 plus positive 5 equals positive 12.*

To add -5 to -7, begin at 7 to the left and count 5 more toward the left, arriving at -12. Thus, $-7 + -5 = -12$, read: *Negative 7 plus negative 5 equals negative 12.*

To add -5 to +7, begin at 7 to the right and count 5 toward the left, arriving at +2. Thus, $+7 + -5 = +2$, read: *Positive 7 plus negative 5 equals positive 2.*

To add +5 to -7, begin at 7 to the left and count 5 toward the right, arriving at -2. Thus, $-7 + +5 = -2$, read: *Negative 7 plus positive 5 equals negative 2.*

The addition of positive and negative numbers is further explained in the following:

50. **Illustrative Problems.** 1. If a man gains \$1500 and then loses \$800, what is the net result? *Answer*, \$700 gain.

In this case the result is obtained by subtracting 800 from 1500. Yet this is not really a problem in subtraction but in addition. That is, we are not asking for the difference between \$1500 gain and \$800 loss, but for the *net result* when the gain and the loss are taken together, or the *sum* of the profit and the loss. Hence, we say \$1500 gain + \$800 loss = \$700 gain, or using positive and negative signs,

$$+1500 + -800 = +700.$$

2. The assets of a commercial house are \$250,000, and the liabilities are \$275,000. What is the net financial status of the house? *Answer*, \$25,000 net liabilities.

Thus,

$$\$250,000 \text{ assets} + \$275,000 \text{ liabilities} = \$25,000 \text{ net liabilities.}$$

Or $+250,000 + -275,000 = -25,000.$

3. The thermometer rises 18 degrees and then falls 28 degrees. What direct change in temperature would produce the same result? *Answer*, 10 degrees fall.

Thus, $18^\circ \text{ rise} + 28^\circ \text{ fall} = 10^\circ \text{ fall.}$

Or $+18 + -28 = -10.$

4. A man travels 700 miles east and then 400 miles west. What direct journey would bring him to the same final destination? *Answer*, 300 miles east.

Thus, $700 \text{ miles east} + 400 \text{ miles west} = 300 \text{ miles east.}$

Or $+700 + -400 = +300.$

EXERCISES

x Perform the following additions

1. $-3 + +4.$

5. $+10 + -10.$

9. $-20 + -10.$

2. $-7 + -3.$

6. $+8 + -4.$

10. $+12 + -12.$

3. $+8 + -16.$

7. $-8 + +17.$

11. $-8 + +8.$

4. $-5 + +5.$

8. $+5\frac{1}{2} + -3\frac{1}{4}.$

12. $-9 + -10.$

The preceding exercises illustrate

Principle VII

✓ 51. Rule. *To add two numbers with like signs, find the sum of their absolute values, and prefix to this their common sign.*

To add two numbers with opposite signs, find the difference of their absolute values, and prefix to this the sign of that one whose absolute value is the greater.

In case their absolute values are equal, their sum is zero.

52. Definition. The sum of two signed numbers thus obtained is called their **algebraic sum**.

Hereafter *addition* will mean finding the *algebraic sum*.

53. Signed numbers find application in any situation where *opposite qualities* of the kind here considered are present. Besides those already mentioned, other instances occur in the applications below:

APPLICATIONS OF SIGNED NUMBERS

1. A balloon which exerts an upward pull of 460 pounds is attached to a car weighing 175 pounds. What is the net upward or downward pull? Express this as a problem in addition, using positive and negative numbers.

Solution. 460 lb. upward pull plus 175 lb. downward pull equals 285 lb. net upward pull. Using positive numbers to represent upward pull and negative numbers to represent downward pull, this equation becomes

$$+460 + -175 = +285.$$

In each of the following translate the solution into the language of algebra by means of signed numbers as in Ex. 1.

2. A 450-pound weight is attached to a balloon which exerts an upward pull of 600 pounds. What is the net upward or downward pull?

3. A man's property amounts to \$45,000 and his debts to \$52,000. What is his net debt or property?

4. The assets of a bankrupt firm amount to \$245,000 and the liabilities to \$325,000. What are the net assets or liabilities?

5. A man can row a boat at the rate of 6 miles per hour. How fast can he proceed against a stream flowing at the rate of $2\frac{1}{2}$ miles per hour? 7 miles per hour?

6. A steamer which can make 12 miles per hour in still water is running against a current flowing 15 miles per hour. How fast and in what direction does the steamer move?

7. A dove which can fly 40 miles per hour in calm weather is flying against a hurricane blowing at the rate of 60 miles per hour. How fast and in what direction is the dove moving?

8. If of two partners, one loses \$1400 and the other gains \$3700, what is the net result to the firm?

9. A man's income is \$2400 and his expenses \$1500 per year. What is the net result for the year?

10. A man loses \$800 and then loses \$600 more. What is the combined loss? Indicate the result as the sum of two negative numbers.

AVERAGES OF SIGNED NUMBERS

54. Half the sum of two numbers is called their *average*. Thus 6 is the average of 4 and 8. Similarly, the average of three numbers is one-third of their sum, and in general the average of n numbers is the sum of the numbers divided by n .

Find the average of each of the following sets:

1. 10, 12, 14, 16, 18.

3. 7, 9, 11, 13, 15.

2. 5, 9, 20, 30, 3.

4. 7, 10, 21, 29, 30.

The average gain or loss per year for a given number of years is the *algebraic sum* of the yearly gains and losses divided by the number of years.

Illustrative Problem. A man lost \$400 the first year, gained \$300 the second, and gained \$1000 the third. What was the average loss or gain?

$$\text{Solution. } \frac{-400 + +300 + +1000}{3} = \frac{+900}{3} = +300.$$

That is, the average gain is \$300.

PROBLEMS

1. Find the average of \$1800 loss, \$3100 loss, \$6800 gain, \$10,800 loss, and \$31,700 gain.

2. Find the average of \$180 gain, \$360 loss, \$480 loss, \$100 gain, \$700 gain, \$400 gain, \$1300 loss, \$300 gain, \$4840 gain, and \$12,000 gain.

Find the average yearly temperatures at the following places, the monthly averages having been recorded as given below :

3. For New York City: $+29^{\circ}$, $+33^{\circ}$, $+39^{\circ}$, $+46^{\circ}$, $+53^{\circ}$, $+63^{\circ}$, $+67^{\circ}$, $+67^{\circ}$, $+61^{\circ}$, $+52^{\circ}$, $+47^{\circ}$, $+41^{\circ}$.

4. For St. Vincent, Minnesota: -5° , 0° , $+15^{\circ}$, $+35^{\circ}$, $+55^{\circ}$, $+60^{\circ}$, $+66^{\circ}$, $+63^{\circ}$, $+55^{\circ}$, $+40^{\circ}$, $+22^{\circ}$, $+5^{\circ}$.

5. For Nerchinsk, Siberia: -23° , -13° , -10° , $+35^{\circ}$, $+55^{\circ}$, $+70^{\circ}$, $+70^{\circ}$, $+64^{\circ}$, $+50^{\circ}$, $+30^{\circ}$, $+5^{\circ}$, -15° .

SUBTRACTION OF SIGNED NUMBERS

55. Definition. Subtraction is the process of finding the number which added to a given number called **subtrahend** produces another given number called **minuend**. The number thus found is called the **difference** or **remainder**.

E.g. We say $8 - 5 = 3$, because $5 + 3 = 8$.

Signed numbers may be subtracted by a simple application of this definition.

Ex. 1. $+8 - +5 = +3$, because $+5 + +3 = +8$.

Ex. 2. $-8 - -5 = -3$, because $-5 + -3 = -8$.

Ex. 3. $-8 - +5 = -13$, because $+5 + -13 = -8$.

Ex. 4. $+8 - -5 = +13$, because $-5 + +13 = +8$.

Signed numbers may be subtracted by counting on the number scale in a direction opposite to that indicated by the sign of the subtrahend.

Thus to subtract $+5$ from $+8$ begin with $+8$ and count 5 units to the left, and to subtract -5 from $+8$ begin with $+8$ and count 5 units to the right.

56. A short rule for subtraction. Since $+8 - -5 = +13$, and since $+8 + +5 = +13$, it follows that *subtracting* -5 from $+8$ gives the same result as *adding* $+5$ to $+8$. Similarly, $-8 - -5 = -3$ and $-8 + +5 = -3$.

Hence, *subtracting a negative number* is equivalent to *adding a positive number* of the same absolute value.

Since $+8 - +5 = +3$, and since $+8 + -5 = +3$, it follows that subtracting $+5$ from $+8$ gives the same result as *adding* -5 to $+8$. Similarly, $-8 - +5 = -13$ and $-8 + -5 = -13$.

Hence, *subtracting a positive number* is equivalent to *adding a negative number* of the same absolute value.

These statements are illustrated by such facts as: *Removing a debt* is equivalent to *adding property* and *removing property* is equivalent to *adding debt*.

Perform the following subtractions by changing the sign of the subtrahend and adding:

- | | | |
|------------------|-------------------|-------------------|
| 1. $-5 - -2$. | 6. $-32 - +34$. | 11. $-16 - +4$. |
| 2. $-4 - +1$. | 7. $-52 - -32$. | 12. $+13 - -20$. |
| 3. $-5 - +2$. | 8. $-16 - -12$. | 13. $-8 - -7$. |
| 4. $+3 - -5$. | 9. $+37 - +50$. | 14. $-19 - +14$. |
| 5. $+57 - -32$. | 10. $-23 - +57$. | 15. $-24 - -19$. |

The preceding exercises illustrate

Principle VIII

57. Rule. *To subtract one signed number from another signed number, change the sign of the subtrahend and then add it to the minuend.*

The change in the sign of the subtrahend may be made *mentally* without rewriting the problem. The results are to be checked by showing that the difference added to the subtrahend equals the minuend.

58. Subtraction always possible. In arithmetic subtraction is possible only when the subtrahend is less than or equal to the minuend.

E.g. In arithmetic we cannot subtract 5 from 2 since there is no positive number which added to 5 gives 2.

However, by means of negative numbers we can as easily perform the subtraction, 2 minus 5, as 5 minus 2.

Thus, $2 - 5 = -3$, since $-3 + 5 = 2$.

EXERCISES

* Perform the following subtractions:

- | | | |
|------------------|-------------------|-------------------|
| 1. $-10 - -5$. | 7. $+6 - -14$. | 13. $-78 - -37$. |
| 2. $-15 - +5$. | 8. $+7 - -9$. | 14. $+57 - +84$. |
| 3. $+20 - -15$. | 9. $-11 - +6$. | 15. $-48 - -31$. |
| 4. $+11 - +3$. | 10. $-21 - -6$. | 16. $-39 - -95$. |
| 5. $-11 - +5$. | 11. $+93 - +22$. | 17. $-91 - -3$. |
| 6. $-17 - -20$. | 12. $+17 - -13$. | 18. $-38 - +74$. |

59. Double use of the Signs $+$ and $-$. In § 46, we agreed that when no sign of quality is written, the sign $+$ is understood. Hence we may write:

$$+8 + +5 = 8 + 5. \quad (1)$$

$$+8 - +5 = 8 - 5. \quad (2)$$

By Principle VII, we have

$$+8 + -5 = +8 - +5 = 8 - 5. \quad (3)$$

By Principle VIII, we have

$$+8 - -5 = +8 + +5 = 8 + 5. \quad (4)$$

These examples show how we may dispense with the special signs of quality $+$, or $-$, as follows:

1. Positive numbers are written without any sign indicating quality except where special emphasis is desired, in which case the sign $+$ is used.

2. A negative number when standing alone is preceded by the sign $-$. Thus -5 is written -5 .

3. When a negative number is combined with other numbers, its quality is indicated by the sign $-$ with a parenthesis inclosing it.

Thus $8 + -5$ is written $8 + (-5)$,
and $8 - -5$ is written $8 - (-5)$.

But in such cases it is customary to apply Principles VII and VIII and write at once $8 - 5$ instead of $8 + (-5)$ and $8 + 5$ instead of $8 - (-5)$.

60. It is clear from the examples in this chapter that signed numbers are needed to represent actual conditions in life, as in case of the thermometer. While from now on such numbers will be distinguished, in accordance with universal custom, by the signs, $+$, $-$, it should be understood that each of these signs is thus made to represent either one of two entirely different things, namely, an *operation* or a *quality*.

However, after we acquire some understanding of the matter, this double use of the signs seldom leads to any confusion, since we can always tell from the context which use is meant. For example, in $5-3$, the sign $-$ means *subtraction*, while in $x=-3$ it means *negative*.

But for the sake of avoiding confusion at the outset, and to make clear that a negative number is not necessarily a *subtrahend* and that a positive number is not necessarily an *addend*, we have up to this time used the special signs $+$, $-$, which could be readily distinguished from the signs of addition and subtraction, $+$, $-$.

EXERCISES

Perform the following indicated operations :

1. $9 - (-4)$.
2. $-10 + (-3)$.
3. $-10 + 6$.
4. $12 - 4$.
5. $5 - 8 - 14$.
6. $-7 + 8 - 18$.
7. Find the value of $a + b$ if (1) $a=4$, $b=-5$; (2) $a=-2$, $b=-7$; (3) $a=-6$, $b=8$; (4) $a=6$, $b=10$.
8. Find the value of $a - b$ if (1) $a=8$, $b=8$; (2) $a=-3$, $b=-7$; (3) $a=4$, $b=-9$; (4) $a=-3$, $b=6$.

Solve the following equations :

9. $x + 8 = 4$. *Suggestion.* Subtract 8 from each member.
10. $x + 3 = 7$.
11. $x - 9 = 1$.
12. $3 + x = 0$.
13. $x + 13 = 7$.
14. $-4 + x = -9$.
15. $-5 + x = 4$.
16. $-5 + x = 12$.
17. $-9 + x = -18$.
18. $-35 + x = 17$.
19. $17 + x = -35$.
20. $x - 14 = -18$.
21. $x - 25 = 16$.

MULTIPLICATION OF SIGNED NUMBERS

61. The multiplication of signed numbers is illustrated by the following problems:

Illustrative Problem. A balloonist, just before starting, makes the following preparations: (a) He adds 9000 cubic feet of gas with a lifting power of 75 pounds per thousand cubic feet. (b) He takes on 8 bags of sand, each weighing 15 pounds. How does each of these operations affect the buoyancy of the balloon?

Solution. (a) A lifting power of 75 lb. is indicated by $+75$, and adding such a power 9 times is indicated by $+9$. Hence, $+9 \cdot (+75) = +675$, or 675 the total lifting power added.

(b) A weight of 15 lb. is indicated by -15 , and adding 8 such weights is indicated by $+8$. Since the total weight added is 120 lb., we have $+8 \cdot (-15) = -120$.

Illustrative Problem. During the course of his journey this balloonist opens the valve and allows 2000 cubic feet of gas to escape, and later throws overboard 4 bags of sand. What effect does each of these operations produce on the balloon?

Solution. (a) The gas, being a lifting power, is positive, but the removal of 2000 cubic feet of it is indicated by -2 , and the result is a depression of the balloon by 150 lb.; that is, $-2 \cdot (+75) = -150$.

(b) The removal of 4 weights is indicated by -4 , but the weights themselves have the negative quality of downward pull. Hence to remove 4 weights of 15 lb. each is equivalent to increasing the buoyancy of the balloon by 60 lb.; that is, $-4 \cdot (-15) = +60 = 60$.

62. These illustrations of multiplying signed numbers are natural extensions of the process of multiplication in arithmetic.

E.g. Just as $3 \cdot 4 = 4 + 4 + 4 = 12$, so $3 \cdot (-4) = -4 + (-4) + (-4) = -12$, and since $3 \cdot 4$ is the same as $+3 \cdot (+4)$, we write $+3 \cdot +4 = +12 = 12$.

Again, just as we take the multiplicand *additively* when the multiplier is a positive integer, so we take it *subtractively* when the multiplier is a negative integer.

E.g. $-3 \cdot (+4)$ means to subtract $+4$ three times; that is, to subtract $+12$. But to subtract $+12$ is the same as to add -12 . Hence, $-3 \cdot (+4) = (-12)$. Again, $-3 \cdot (-4)$ means to subtract -4 three times; that is, to subtract -12 . But to subtract -12 is the same as to add $+12$. Hence, $-3 \cdot (-4) = +12 = 12$.

EXERCISES AND PROBLEMS

Explain the following indicated multiplications and find the product in each case:

- | | | |
|-----------------------|-----------------------|------------------------|
| 1. $-3 \cdot (-10)$. | 4. $-7 \cdot (-8)$. | 7. $-5 \cdot (-48)$. |
| 2. $10 \cdot (-3)$. | 5. $12 \cdot (-21)$. | 8. $-25 \cdot (-16)$. |
| 3. $50 \cdot (-5)$. | 6. $-27 \cdot (-6)$. | 9. $-8 \cdot 34$. |

10. A man gained \$212 each month for 5 months, then lost \$175 per month for 3 months. Express his net gain or loss as the sum of two products.

11. A raft is made of cork and iron. What effects are produced upon its floating qualities by the following changes?
 (a) Adding 4 braces, each weighing (under water) 5 pounds.
 (b) Removing 3 pieces of cork, each capable of sustaining 3 pounds.
 (c) Adding 10 pieces of cork, each capable of sustaining 7 pounds.

The preceding exercises illustrate

Principle IX

63. Rule. *If two numbers have the same sign, their product is positive; if they have opposite signs, their product is negative.*

In applying this principle observe that the sign of the product is obtained quite independently of the absolute value of the two factors.

E.g. $\frac{1}{4} \cdot (-5) = -(\frac{1}{4} \cdot 5) = -3\frac{1}{4}$; $-12 \cdot (-3.5) = +42 = 42$.

64. The product of several signed numbers is found as illustrated in the following:

$-2 \cdot 5 \cdot (-3) \cdot (-4) \cdot 6 = -10 \cdot (+3) \cdot (-4) \cdot 6 = 30 \cdot (-4) \cdot 6 = -120 \cdot 6 = -720$. That is, the first two factors are multiplied together, then this product by the next factor, and so on, until all the factors are multiplied.

Since the product of all positive factors is positive, the final sign depends upon the number of negative factors. If this number is *even*, the product is positive; if it is *odd*, the product is negative.

E.g. If there are 5 negative factors, the product is negative; if there are 6, it is positive.

In the following exercises determine the sign of the product before finding its absolute value.

EXERCISES

1. $-4 \cdot 3 \cdot (-6) \cdot (-7)$.
2. $-2 \cdot (-3) \cdot (-5) \cdot 3$.
3. $-5 \cdot [-3 + (-7)]$.
4. $-5 \cdot (-4) \cdot 3 \cdot (-2)$.
5. $8 \cdot (-9) \cdot (-1) \cdot (-2)$.
6. $-50 \cdot (-20) \cdot (-30) \cdot (-40)$.
7. $-2 \cdot (-3) \cdot (-4) \cdot m \cdot (-n) \cdot p \cdot (-q)$.

DIVISION OF SIGNED NUMBERS

65. In arithmetic we test the correctness of division by showing that the quotient multiplied by the divisor equals the dividend.

E.g. $27 \div 9 = 3$, because $9 \cdot 3 = 27$.

Hence *division* may be defined as the process of finding one of two factors when their product and the other factor are given.

The given product is the **dividend**, the given factor the **divisor**, and the factor to be found is the **quotient**.

This definition also applies to the division of signed numbers. In dividing signed numbers, however, we must determine the *sign* of the quotient as well as its *absolute value*.

E.g. $-42 \div (+6) = -7$, because $-7 \cdot (+6) = -42$;
also $-42 \div (-6) = +7$, because $+7 \cdot (-6) = -42$.

So in every case the test is:

$$\text{Quotient} \times \text{Divisor} = \text{Dividend}.$$

In like manner perform the following, and check as above:

1. $-25 \div 5$.

4. $-9rs \div +3$.

2. $-ab \div a$.

5. $75y \div (-15)$.

3. $5xy \div (-x)$.

6. $-121x \div 11$.

The preceding exercises illustrate

Principle X

66. Rule. *The quotient of two signed numbers is positive if the dividend and divisor have like signs, negative if they have opposite signs.*

EXERCISES

Perform the following indicated divisions, and check by multiplying quotient by divisor.

1. $\frac{-28}{7}$.

5. $\frac{-75}{5}$.

9. $\frac{4 \cdot (-9)}{-3}$.

2. $\frac{-42}{-6}$.

6. $\frac{-16}{-1}$.

10. $\frac{-3 \cdot 8}{-4}$.

3. $\frac{51}{-17}$.

7. $\frac{-49}{1}$.

11. $\frac{100 \cdot (-99)}{-25}$.

4. $\frac{-21}{-3}$.

8. $\frac{64}{-1}$.

12. $\frac{3 \cdot (-4) \cdot (-6) \cdot 8}{-3}$.

13. A man lost \$300, \$500, and \$700 during three consecutive months. Express his average monthly loss as a quotient.

14. During five consecutive days the maximum temperature was -5° , -8° , -10° , -4° , -6° respectively. Find the average of these high marks.

15. A trader lost \$250 in each of three months and gained \$75 during each of the four succeeding months. Find the average gain or loss for the seven months.

67. While Principles I-V were studied in connection with unsigned, or arithmetic numbers only, it is now very important to note that they all apply to *signed* numbers as well.

In the statement of these principles the word *number* will from now on be understood to refer either to the ordinary numbers of arithmetic or to the signed numbers, as occasion may require. It should also be noticed that the numbers of arithmetic are used as freely in algebra as in arithmetic. It is only when we wish to distinguish them from negative numbers that they are called positive numbers.

The **number system of algebra**, so far as we have studied it consists of the *numbers of arithmetic* together with the *negative numbers*.

Historical Note. The Hindus appear to have had quite clear notions of a purely "negative number" as distinct from a number to be subtracted. They recognized the difference between positive and negative numbers by attaching to one the idea of debt and to the other that of assets, or by letting them represent distances in opposite directions. The Arabs, however, failed to understand the negative numbers and did not include them in the algebra which they brought to Europe. (See page 19.) With unimportant exceptions, until the beginning of the seventeenth century, mathematicians dealt exclusively with positive numbers.

The negative numbers were brought permanently into mathematics by René Descartes. (See pages 19 and 114.) Trying to number all the points of a complete straight line, Descartes was compelled to start at a point and number in both directions. Then it became convenient to distinguish the numbers on the two sides of this starting-point as positive and negative, respectively.

Sir Isaac Newton (1642-1727) was the first to let a letter stand for any number, negative as well as positive. In such a formula as $a(b + c) =$

$ab + ac$, the predecessors of Newton would restrict the letters to represent any positive numbers, while Newton regarded the letters as representing *any numbers whatever* either positive or negative. This was of very great importance, since it greatly reduced the number of formulas required.

Negative numbers appeared "absurd" or "fictitious" to mathematicians until they hit upon a visual or graphical representation of them. Cajori in his history of elementary mathematics says: "Omit all illustrations by lines, thermometers, etc., and negative numbers will be as obscure to modern students as they were to the early algebraists." From the experience of the early mathematicians it would appear that if the pupil wishes to really understand positive and negative numbers, he must study with care applications such as are given in the first part of this chapter.

INTERPRETATION AND USE OF NEGATIVE NUMBERS

68. In solving a problem, a negative result may have a natural interpretation or it may indicate that the conditions of the problem are impossible.

A similar statement holds in reference to fractional answers in arithmetic. For example, if we say there are twice as many girls as boys in a schoolroom and 35 pupils in all, the number of boys would be $35 \div 3 = 11\frac{2}{3}$, which indicates that the conditions of the problem are impossible.

69. **Illustrative Problem.** The crews on three steamers together number 94 men. The second has 40 more than the first, and the third 20 more than the second. How many men in each crew?

Solution. Let n = number of men in first crew.

Then, $n + 40$ = number of men in second crew,

and $n + 40 + 20$ = number of men in third crew.

Hence, $n + n + 40 + n + 40 + 20 = 94$,

and $3n + 100 = 94$.

$$3n = -6.$$

$$n = -2.$$

Here the negative result indicates that the conditions of the problem are *impossible*.

70. Illustrative Problem. A real estate agent gained \$8400 on four transactions. On the first he gained \$6400, on the second he lost \$2100, on the third he gained \$5000. Did he lose or gain on the fourth transaction?

Solution. Since we do not know whether he gained or lost on that transaction, we represent the unknown number by n , which may be positive or negative, as will be determined by the solution of the problem.

$$\text{Thus we have} \quad 6400 + (-2100) + 5000 + n = 8400. \quad (1)$$

$$\text{Hence, by VII, F,} \quad 9300 + n = 8400. \quad (2)$$

$$\text{By S,} \quad n = 8400 - 9300. \quad (3)$$

$$\text{By VIII,} \quad n = -900. \quad (4)$$

In this case the negative result indicates that there was a *loss* on the fourth transaction.

PROBLEMS

In the following problems give the solutions in full and state all principles used, together with the interpretation of the results:

1. A man gains \$2100 during one year. During the first three months he loses \$125 per month, then gains \$500 per month during the next five months. What is the gain or loss per month during the remaining four months?

2. A man rowing against a swift current goes 9 miles in 5 hours. The second hour he goes one mile less than the first, the third two miles more than the second, and the fourth and fifth each one mile more than the third hour. How many miles did he go during each of the five hours?

3. There are three trees the sum of whose heights is 108 feet. The second is 40 feet taller than the first, and the third is 30 feet taller than the second. How tall is each tree?

Find the average yearly temperature at each of the following places, the average monthly temperatures being as here given:

4. Port Conger, off the northwest coast of Greenland: -37° , -43° , -32° , -15° , 14° , 18° , 35° , 34° , 25° , 4° , -17° , -30° .

5. Franz Joseph's Land: -20° , -20° , -10° , 0° , 15° , 30° , 35° , 30° , 20° , 10° , 0° , -10° .

6. North Central Siberia: -60° , -50° , -30° , 0° , 15° , 40° , $+40^{\circ}$, 35° , 30° , 0° , -30° , -50° .

+ 7. A merchant gained an average of \$2800 per year for 5 years. The first year he gained \$3000, the second \$1500, the third \$4000, and the fourth \$2400. Did he gain or lose and how much during the fifth year?

8. A certain business shows an average gain of \$4000 per year for 6 years. During the first 5 years the results were: \$8000 loss, \$10,000 gain, \$7000 gain, \$3000 gain, and \$12,000 gain. Find the loss or gain during the sixth year.

9. A commercial house averaged \$15,000 gain for 6 years. What was the loss or gain the first year if the remaining years show: \$8000 gain, \$24,000 gain, \$2000 loss, \$20,000 gain, and \$50,000 gain, respectively?

REVIEW QUESTIONS

1. Name several pairs of opposite qualities all of which are conveniently described by the words *positive* and *negative*. What symbols are used to replace these words when applied to numbers?

2. When loss is added to profit, is the profit increased or decreased? What algebraic symbols may be used to distinguish the numbers representing profit and loss?

3. On the number scale indicate what is meant by $+2$; by -2 . Indicate what is meant by the sign $+$ in $5+2$; by the sign $-$ in $5-2$; by the sign $-$ in $x=-2$.

4. Why do we call positive and negative numbers signed numbers? What is meant by the absolute value of a number?

5. State Principle VII in full.

6. How is the correctness of subtraction tested in arithmetic? Is the same test applicable to subtraction in algebra?

7. Illustrate the subtraction of positive and negative numbers by an example involving profit and loss.

8. Show by counting on the number scale that the result of subtraction gives the *distance* from subtrahend to minuend and that the sign of the remainder shows the *direction* from subtrahend toward the minuend. For example, use $8 - (-5)$ and $-8 - (+5)$ to illustrate this.

9. How do negative numbers make subtraction possible in cases where it is impossible in arithmetic?

10. What is a convenient rule for subtracting signed numbers? State Principle VIII.

11. Write an equation whose solution is a negative number.

12. Give an example in which positive and negative numbers are multiplied. State Principle IX.

13. Define division. How do we obtain the law of signs in division? State Principle X. What is the test of the correctness of division?

14. Explain how one set of signs $+$ and $-$ can be used to indicate both quality and operation.

15. By means of Principles VII, VIII, IX, and X, simplify the expressions, $a + b$, $a - b$, $a \cdot b$, $\frac{a}{b}$, after substituting in each various positive and negative values of a and b .

16. Add Principles VII, VIII, IX, and X to the list which you made in Chapters I and II. It is absolutely necessary that you remember the rules stated in these principles. Any short phrases that will assist you in this are of value. For instance the following:

VII. *In addition, positive and negative numbers tend to cancel each other. The common sign or the sign of the numerically greater is the sign of the result.*

VIII. *In subtraction, change the sign of the subtrahend and add.*

IX. *In multiplication, like signs give $+$ and unlike signs $-$.*

X. *In division, like signs give $+$ and unlike signs give $-$.*

DRILL EXERCISES

In solving the following equations perform the required additions or subtractions by the direct method suggested in § 38.

1. $5x + 7 - 2x = 2x + 9$.
2. $3n + 2(n + 4) = 4n + 14$.
3. $8 + 6x + 3(2 + x) = 5x + 26$.
4. $4(x + 3) + 2(3x + 1) = 5(x + 2) + 2(x + 3) + 19$.
5. $7(x + 1) + 3(2x + 3) = 4(x + 5) + 7(x + 2) - 8$.
6. $7(2x - 3) + 5(4x - 1) = 3(x + 1) + 2$.

Perform the following indicated operations:

7. $-8 - (-9) - (+7) + 8$.
8. $16 + (-18) - (+2) + 4$.
9. $12x + (-7x) - (-3x) + 2x$.
10. $34n + (-30n) - (-7n)$.
11. $8 \cdot (-3) \cdot (-1)$.
14. $(-12) \div [(-2) + (-2)]$.
12. $[48 \div (-6)] \div (-1)$.
15. $(-42) \cdot (-2) \div (-7)$.
13. $3 \cdot (-2)(-3)(-1)$.
16. $[(7ab) \div (-a)] \cdot (-3)$.
17. What is meant by the average of several numbers?
18. Find the average of 20, 16, 8, 4, 0, -8, -12.
19. Find the average of -8, -32, 14, 26, -40.

Solve the following:

20. $x + 6 = 4$.
23. $3x - 8 = -16$.
21. $3x + 12 = 6$.
24. $2(x + 6) = 3(x + 5)$.
22. $8 + x = 4$.
25. $2x + 4 = 3x + 8$.

26. If $-n$ represents a negative integer, how do you represent the next integer to the right on the number scale? the next to the left?

27. If $-2n$ represents a negative even integer, how do you represent the next even integer to the right? the next to the left?

CHAPTER IV

POLYNOMIALS

71. We have found that the solution of problems leads us to build **number expressions** out of single number symbols.

E.g. If x is a number representing my age in years, then $2(x - 10)$ is double the number representing my age 10 years ago.

Also $2[(x - 10) + (x + 15)]$ is the number representing twice the sum of my ages 10 years ago and 15 years hence.

Number expressions are now to be studied more in detail.

72. **Definition.** A number expression composed of parts connected by the signs $+$ and $-$ is called a **polynomial**. Each of the parts thus connected, together with the sign preceding it is called a **term**.

E.g. $5a - 3xy - \frac{2}{3}rt + 99$ is a polynomial whose terms are $5a$, $-3xy$, $-\frac{2}{3}rt$, and $+99$. The sign $+$ is understood before $5a$.

Likewise $3x + \frac{2}{x} + \frac{4}{y}$ is a polynomial whose terms are $3x$, $\frac{2}{x}$, and $\frac{4}{y}$.

NOTE.—The word *polynomial* is sometimes used in a more restricted sense in higher mathematics.

73. **Definitions.** A polynomial of two terms is called a **binomial**; one of three terms is called a **trinomial**. A term taken by itself is called a **monomial**.

E.g. $5a - 3xy$ is a binomial; $5a - 3xy - \frac{2}{3}rt$ is a trinomial whose terms are the monomials $5a$, $-3xy$, $-\frac{2}{3}rt$.

According to this definition $x + (b + c)$ may be called a binomial though it is equivalent to the trinomial $x + b + c$.

In this case x is called a **simple term** and $(b + c)$ a **compound term**. Likewise we may call $3t + 4x - 5(a + b)y$ a trinomial having the simple terms $3t$, $+4x$, and the compound term $-5(a + b)y$.

74. Definition. Two terms which have a factor in common are called **similar with respect to that factor**.

E.g. $5a$ and $-3a$ are similar with respect to a ; $-3xy$ and $-7x$ are similar with respect to x ; $5a$ and $-5b$ are similar with respect to 5 ; $7abc$ and $-\frac{1}{2}abc$ are similar with respect to abc .

Similar terms may be combined by Principle I.

E.g. $5a - 3a = (5 - 3)a = 2a$; $-3xy - 7x = -x(3y + 7)$; $5a - 5b = 5(a - b)$.

EXERCISES

Select the common factor and combine the similar terms in each of the following :

- | | |
|--------------------------|--------------------------|
| 1. $7x - 5x + 4x$. | 8. $3ab - 2bc + 5bd$. |
| 2. $3a - 2a + 4a$. | 9. $7ax + 3bx + 12cx$. |
| 3. $7a + 2a - 5x + 7x$. | 10. $5ax + 3ax - 2ax$. |
| 4. $3x - 2x + 4x + 2x$. | 11. $2ar + 2br - 2cr$. |
| 5. $8r + 3r + 2r - 5r$. | 12. $11rs - 2st + 4as$. |
| 6. $9t - 3t + 4t - 3t$. | 13. $6ab + 7ac - ad$. |
| 7. $ax + bx + cx$. | 14. $4xy - 3yz - 5wy$. |

ADDITION AND SUBTRACTION OF POLYNOMIALS

Addition of Polynomials. In adding polynomials we use

Principle XI

75. Rule. *If two or more terms are to be added, they may be arranged and combined in any desired order.*

The truth of this principle may be seen from simple examples :

Thus, $2 + 3 + 5 = 3 + 2 + 5 = 2 + (3 + 5) = (3 + 2) + 5 = 13$.

Also, $8 + (-2) + 6 = -2 + 8 + 6 = -2 + (8 + 6) = 12$.

Make other examples to illustrate this principle.

Historical Note. The fundamental character of Principle XI was first recognized about one hundred years ago. The principle as here given combines in one statement two laws of algebra : (1) the **associative law**, first so called by F. S. Servois (1814) ; (2) the **commutative law**, first so called by Sir William Hamilton.

76. In adding polynomials the work may be conveniently arranged by placing the terms in columns, each column consisting of terms which are similar. This is permissible by Principle XI.

Ex. Add $5x - 6y + 4z + 5at$, $-3x + 11y - 16z - 9bt$, and $-7y + 8z$.

Arranging as suggested and applying Principles I and VII we have

$$\begin{array}{r}
 5x - 6y + 4z + 5at \\
 - 3x + 11y - 16z - 9bt \\
 \quad - 7y + 8z \\
 \hline
 2x - 2y - 4z + (5a - 9b)t
 \end{array}$$

$5x$ and $-3x$ are similar with respect to their common factor x . Hence, by Principle I we add the other factors 5 and -3 , obtaining $(5 - 3)x = 2x$.

Likewise we add $+5at$ and $-9bt$ with respect to the common factor t , obtaining $(5a - 9b)t$. In the second column the sum is $(-6 + 11 - 7)y = -2y$, and in the third column the sum is $(+4 - 16 + 8)z = -4z$.

Check by giving convenient values to x, y, z, t, a , and b .

EXERCISES IN ADDITION

1. Add $7b - 3c + 2d$; $-2b + 8c - 13d$.
2. Add $6x - 3y + 4t - 7z$; $x - 5y - 3t$; $4x - 4y + 8t$; $8x + 2y - 3t$.
3. Add $7a - 4x + 12z$; $8a - 3x + 2z$; $2a + 4x - 3z$; $5a - 2x - 4z$.
4. Add $5ac + 3bc - 4c + 8b$; $2b + 3c - 2bc - 3ac$; $4b + 4c + bc - ac$; $2bc + 4ac + c$; $3b - 4c$.
5. Add $16xy - 13cd$; $15ab - 2xy$; $34cd - 3xy + 2ab$; $14cd - 3xy - 2ab$.
6. Add $34ax + 4by - 3z$; $2by + 5z$; $3ax - 7by + 5z$; $7ax + 4by - az$.
7. Add $3b + 4cd - 2ae$; $ab - 3cd + 3ae$; $3cd - 2ab$; $4cd - 5ae + 7ab$.

8. Add $7ax - 13by + 5$; $9ax + 8by - 4$; $3by - 12ax$; $4ax + 7by - 9$.

9. Add $5ab - 3 \cdot 67 + 5(x - 1)$; $5 \cdot 67 + 3ab - 2(x - 1)$; $3(x - 1) - 4 \cdot 67 + 2ab$.

10. Add $11(c - 9) + 3(x + y) + 21wu$; $-71wu - 5(x + y) - 13(c - 9)$.

11. Add $5(a + b) - 3(c - d)$; $3(c - d) - 8(a + b)$; $-2(a + b)$; $13(c - d) - 4(a + b)$.

12. Add $3 + 4(c - d) - 5(a - b - c)$; $4(a - b - c) + 5(c - d)$; $3(a - b - c) - 9(c - d) + 12$.

13. Add $(a - b) - 3(c - d) + 4(a - b)$; $5(a - b) + 4(c - d) + 7(c - d) - 9(a - b)$.

14. Add $7(x - y) - 4(x + y) + 4 \cdot 7$; $9(x + y) + 3(x - y) - 9 \cdot 7$; $6(x - y) + 2 \cdot 7 - 3(x + y)$.

15. Add $3(x - 5) + 4(c + b) + 3(x - y)$; $8(c + b) - 5(x - y) + 8(x - 5)$; $7(c + b) - 4(x - y)$; $3(x - y) + (x - 5)$.

16. Add $16(a + b - c) - 3(x - y) + 2(a - b)$; $2(x - y) - 3(a - b) + (a + b - c)$; $7(a - b) + 4(x - y) - 8(a + b - c)$.

17. Add $6(a - b) - 5(x + y) + 7(x - z) - 4abc$; $7(x - z) - 9(x + y) + (a - b) + 2abc$; $11(a - b) + 10abc + 3(x - z) + 8(x + y)$.

77. Subtraction of Polynomials. Since subtraction is performed by adding the subtrahend, with its sign changed, to the minuend, Principle XI permits us to arrange the terms in any desired order as in addition.

This is illustrated as follows:

From $15ab - 17xy + 11rt$ subtract $-5ab + 4xy - 5nt$.

Arranging as on page 58 and applying Principles I and VIII:

$$\begin{array}{r} 15ab - 17xy + 11rt \\ - 5ab + 4xy - 5nt \\ \hline 20ab - 21xy + 11rt - 5nt \end{array}$$

As suggested in § 57, it is sufficient to change the signs of the subtrahend *mentally*, rather than to rewrite them before adding to the minuend.

EXERCISES IN SUBTRACTION

1. From $9x + 3y - 11z$ subtract $-5x + 8y - 3z$.
2. From $12ab - 3cd + 12xy$ subtract $3ab + 2cd - 11xy$.
3. From $9xc + 4ad - 3cz + 5y$ subtract $3y - 3ad + 5cz$.
4. From $13t + 5mx - 5cv$ subtract $2t - 4mx - 3cv$.
5. From $3v - 2w + 5mn - 4xz$ subtract $-v + 5w - 3mn$.
6. From $31b + 4xy + 16ax - 4$ subtract $8b - 5xy - 3ax$.
7. From $4 - 3a - 5xz - 3vy - x$ subtract $7a + 2xz + 4vy$.
8. From $8xy - 3x + 4y$ subtract $-2xy + 13w + 4x - 2y$.
9. From $2ab - 5 + 7v + 13abc$ subtract $3ab + v + 8abc$.
10. From $8cxa - 4yb - 3yc$ subtract $4cxa + 2yb + 4yc - 49$.
11. From $31 \cdot 45 - 7xy$ subtract $12 \cdot 45 + 9xy$.
12. From $3abc - 4 + 2(x + y) - 3xy$
subtract $28 + 4xy - 3(x + y) + 8abc$.
13. From $21 + 9(xy - z) + 3(a + b)$
subtract $8(xy - z) - 8(a + b) + 15$.
14. From $5ax - 3by + 4ax + 5by$ subtract $5by + 3ax + 7by$.
15. From $15 \cdot 48 + 8ab + 49x$ subtract $7 \cdot 48 - 9ab - 14x$.
16. From $19(r - 5s) + 13(5x - 4) + 7(x - y)$
subtract $17(5x - 4) - 5(x - y) - 11(r - 5s)$.
17. From $30 + 14(x - 5yz) - 13(5y - z)$
subtract $32 + 8(5y - z) - 7(x - 5yz)$.
18. From $a(b + c) + 4(m + n) - 16c$
subtract $9(m + n) + 31c - d(b + c)$.
19. From $5(7x - 4) + 3(5y - 3x) + 35$
subtract $56 - 9(7x - 4) + 8(5y - 3x)$.
20. From $(3a + 9b - 12c) \div 3$
subtract $(6a - 12b - 18c) \div 6$.
21. From $(axy + ayz - axz) \div a$
subtract $y(x + z) - 2xz$.

EXERCISES IN ADDITION AND SUBTRACTION

1. Add $5x - 3y - 7r + 8t$, $-7x + 18y - 4r - 7t$, $-20x - 24y + 18r - 15t$, and $13x + 15y + 11r + 6t$.

Check the sum by substituting $x=1$, $y=1$, $r=1$, $t=1$.

2. Add $17a - 9b$, $3c + 14a$, $b - 3a$, $a - 17c$, and $a - 3b + 4c$. Check for $a=1$, $b=2$, $c=3$.

3. Add $2x + 3y - t$, $-6y + 8t$, $-x + y - t$, $-4t + 7x$, and $3y$. Check for $x=2$, $y=3$, $t=1$.

4. Add $17r + 4s - t$, $2t + 3u$, $2r - 3s + 4t$, $5u - 6t$, $7r - 3s + 8u$, and $8r - 2t + 6u$. Check by putting each letter equal to 1; also equal to 2.

5. Add $3h + 2t + 4u$ and $h + 3t + 3u$. Check by putting $h=100$, $t=10$, $u=1$; i.e. $324 + 133 = 457$.

6. Add $4h + 3t + u$ and $3h + 2t + 7u$. Check as in 5.

7. Write 247, 323, 647, 239, and 41, as number expressions like those in Exs. 5 and 6 and then add them.

8. Add 647, 391, 276, and 444 as in Ex. 7.

9. Add $4t - u$, $5t - u$, $6t - u$, $7t - u$, and $8t - u$. Check for $t=10$, $u=1$; also $t=1$, $u=1$.

10. Simplify: $3xyz - 2xyz + 5xyz - 4xyz + xyz - xyz$.

11. Subtract $5a - 3b + 6c$ from $-8a + 7b - 11c$ and check.

12. From $7xy + 8xz + 9yz$ take $17xy - 19xz - 20yz$.

13. From $6x - 3y$ take $8y - 3z$.

14. From $3p - 4q + 8r$ take $7p - 11r + 11q$.

15. From $2x - 3y$ take $5x + 7y + 2a - 3b$.

16. From the sum of $18abc - 27xyz + 13rst$ and $-11abc + 16xyz - 52rst$ take $67rst - 39abc$.

17. To the difference between the subtrahend $15x - 18y + 27z$ and the minuend $117x + 97y - 81z$ add $4x - 6y + 3z$.

18. Add $11(x - y) + 15(a - b)$ and $-20(x - y) - 37(a - b)$ and from the sum subtract $135(x - y) - 213(a - b)$.

NUMBER EXPRESSIONS IN PARENTHESES

78. The sign $+$ before a parenthesis means that each term within is to be added to what precedes, and the sign $-$ means that each term within is to be subtracted from what precedes.

By Principle VII, $a + (+b) = a + b$ and $a + (-b) = a - b$; and by Principle VIII, $a - (+b) = a - b$ and $a - (-b) = a + b$. Hence, we have

Principle XII

79. Rule. *A parenthesis preceded by the sign $+$ may be removed without further change.*

A parenthesis preceded by the sign $-$ may be removed by changing the sign of each term within it.

Note that in each case the sign preceding the parenthesis is also removed after the operation indicated by it has been carried out, and that if no sign is written before the first term in the parenthesis, the sign $+$ is understood.

Remove the parentheses and simplify the following:

$$\begin{aligned}\text{Ex. 1. } 3a + (a - b + 4) - (2a + 3b - 2) \\ = 3a + a - b + 4 - 2a - 3b + 2 = 2a - 4b + 6.\end{aligned}$$

$$\begin{aligned}\text{Ex. 2. } 5(3x + y) - 4(2x - 3y + 2) \\ = 15x + 5y - 8x + 12y - 8 = 7x + 17y - 8.\end{aligned}$$

In Ex. 2 we multiply the terms within the first parenthesis by 5 and those in the second by 4 and then remove the parentheses by Principle XII.

EXERCISES

Remove the symbols of aggregation and simplify:

- $(3x - 2y) - (4x + 3y - 2).$
- $x - y - 2z - (3x + 2y - 7z).$
- $3(a + b + c) - 2(a - b + c).$
- $8(5x - y + 2z) - 11(3x + y - z).$
- $5(7x - 4y) + 9(x - y) - 3(2x + 3y).$
- $8(r - s) + (2r + s) - (r - 2s).$
- $11t + (2t - 1) - (1 - 3t).$

8. $9(r-s) - 3(r+s) + 2(2r-s)$.
9. $3(5x-7y) - (4x-3y+2) - 5y$.
10. $5x - (8-4x+7y) + (5x+3) - (5y+3x-99)$.
11. $-(3a+5b-7c) + (8a-4c) - (9c-4b+4a) - 91a$.
12. $7 - (4-4c+2d-2a) + 31c - (4-2a-5d) - (-8c)$.
13. $(41ab-21c+4) - (36c+15-78ab) + (13c-90ab-8)$.
14. $9by - (4c-8by-13) - 2c - 16 - (34by-12c+8by)$.
- ✓ 15. $6mn + (-9m-7n+14) - 8n + (13mn-17m) + 34mn$.
16. $34ax - (-17ax+42) + 8x - (14a+24ax-7)$.
17. $19 - (+2-7a-4b+11ab) - (-2b+8ab+4a)$.
18. $41by - (4b-13y+17by) - (-5b-17by+13y)$.
19. $39rs - 20s - 19r - (7rs+8s-19r) - (15r-5s-56)$.
20. $x - \{8x - (2y-3x) + (2x-4y)\}$.

Suggestion. First remove the parentheses, then the brace.

21. $a + [a - (b+c) - 2c]$.
22. $a - \{-(a-b) + (3a-2b)\}$.
23. $2x - 3(x-1) - [x - 2(2x-1)]$.
- ✓ 24. $a - \{a + (b-c) - 2(a+b+c)\}$.

80. By the converse of Principle XII terms may be inclosed in a parenthesis with or without change of sign, according as the sign $-$ or $+$ precedes.

E.g. $a + b - c = a + (b - c)$ and $a - b + c = a - (b - c)$.

EXERCISES

In each of the following, place the last three terms in a parenthesis.

1. $x - y + 2 - 5$.
2. $5a + 3b - c + d$.
3. $m - n + p - q$.
4. $5a - 3b + 2c - d$.
5. $7m - 4n - 3p - q$.
- ✓ 6. $8 + 4b - 3c - d$.
7. $5xy - 5x + 3y - 2$.
- ✓ 8. $9ax + 3by - 4cd + 2$.
9. $a - 3b + d - 5c + 8$.
10. $13 - 7a - 3b + 9c$.
11. $19x - 3c + 4e - 18d$.
12. $21ax - 13bx + 6dx - 8e$.

PROBLEMS

See suggestions on solving problems on page 31. Check each result by showing that it satisfies the conditions of the problem.

+ 1. The sum of two numbers is 16. Seven times one is 8 less than 5 times the other. What are the numbers?

2. From a certain number a there is subtracted 3 times the remainder when 8 is subtracted from $2a$. Express the result in terms of a .

3. A man invested a certain sum of money at 5% simple interest. The amount $3\frac{1}{4}$ years later was \$950. What was the investment?

4. A man bought a tract of coal land and sold it a month later for \$93,840. If his gain was at the rate of 24% per annum, what did he pay for the land?

5. The melting point of copper is 250 degrees (Centigrade) lower than 4 times that of lead. Ten times the number of degrees at which lead melts minus twice the number at which copper melts equals 1152. What is the melting point of each metal?

4 ⑥ The Nile is 100 miles more than twice as long as the Danube. Ten times the length of the Danube minus 4 times the length of the Nile equals 3400 miles. How long is each river?

7. The Ganges River is 1800 miles shorter than the Amazon, and the Orinoco is 300 miles shorter than the Ganges. The sum of their lengths is 6900 miles. How long is each?

8. Lead weighs 259 pounds more per cubic foot than cast iron, and 166 pounds more than bronze; while a cubic foot of bronze weighs 807 pounds less than 3 cubic feet of iron. Find the weight per cubic foot of each metal.

9. The world's gold production in 1908 was 29 million dollars less than 3 times that of 1893, and the production in 1900 was 59 million less than twice that of 1893. The production of 1900 and 1908 together amounted to 697 million. How much was produced each year?

PROBLEMS ON THE ARRANGEMENT AND VALUE OF DIGITS

81. If we speak of the number whose 3 digits, in order from left to right, are 5, 3, and 8, we mean $538 = 500 + 30 + 8$. Likewise, the number whose three digits are h , t , and u is written $100h + 10t + u$.

Hence, when letters stand for the digits of numbers written in the decimal notation, care must be taken to multiply each letter by 10, 100, 1000, etc., according to the position it occupies.

Illustrative Problem. A number is composed of two digits whose sum is 6. If the order of the digits is reversed, we obtain a number which is 18 greater than the first number. What is the number?

Solution. Let $x =$ the digit in tens' place.

Then, $6 - x =$ the digit in units' place.

Hence, the number is $10x + 6 - x$. Reversing the order of the digits, we have as the new number $10(6 - x) + x$.

Hence, $10(6 - x) + x = 18 + 10x + 6 - x$.

In each of the examples 1 to 8 below there is a number composed of two digits.

1. The digit in units' place is 2 greater than the digit in tens' place. If 4 is added to the number, it is then equal to 5 times the sum of the digits. What is the number?

2. The digit in tens' place is 3 greater than the digit in units' place. The number is one more than 8 times the sum of the digits. What is the number?

3. The sum of the digits is 9. If the order of the digits is reversed, we obtain a number which is equal to 12 times the remainder when the units' digit is taken from the tens' digit. What is the number?

4. The sum of the digits is 12. If the order of digits is reversed, the number is increased by 18. Find the number.

5. The tens' digit is 2 less than its units' digit. The number is 1 less than 5 times the sum of its digits. What is the number?

6. The digit in units' place is 4 less than that in tens' place. If the order of the digits is reversed, we obtain a number which is 3 less than 4 times the sum of the digits. What is the number?

+ 7. The digit in units' place is 2 less than twice the digit in tens' place. If the order of the digits is reversed, the number is unchanged. What is the number?

8. The digit in tens' place is 12 less than 5 times the digit in units' place. If the order of the digits is reversed, the number is equal to 4 times the sum of the digits. What is the number?

9. A number is composed of three digits. The digit in units' place is 3 greater than the digit in tens' place, which in turn is 2 greater than the digit in hundreds' place. The number is equal to 96 plus 4 times the sum of the digits. What is the number?

DRILL EXERCISES

- ✓ 1. Add $3x + 4y - 3z$, $5x - 2y - z$, and $3y - 5x + 7z$.
2. From $15a + 4b - 13bc$ subtract $3a - 8b + 2bc$.
3. Subtract $7x - 5y - 7a$ from $6x + 5y + 3a$.
4. From $5x - 4y - 9z$ subtract $3x - 8y + 2z$.
5. Add $5a + 3b - 2c$ and $11a - 7b + 8c$.
6. Add $11axy + 13x - 14y$, $2y - 4x$, and $3y + x - 8axy$.
7. $(5x - 3b) + (2x + b) - (4x - 2b - x + 5b)$.
8. Add $19b + 3c$, $2b - 7c$, $2c - 14b$, and $c + 8b$.
9. $-(a - 3b - c) - (2c - a - 5b) + (a - c + b)$.
10. Subtract $2x + 4y + z$ from $13x - 3y - 5z + 8$.
11. $5(x - 7) + 3(14 - x) + 60 = 1 - 10x$.
12. $13(1 - x) - 6(2x - 5) = 80 + 12x$.
13. Add $7x - 3y - 4$, $5x + 2y + 5$, and $3y - 8x - 6$.
14. Add $13a + 4b - 9c$, $2c - 8b - 16a$, and $8a - 5b - 8c$.
15. $8x - [2x + 3(x - 1) - (2x - 3)]$.
16. From $17b - 4a - 2c - 19$ subtract $8c - 5a - 8b + 4$.

17. $3 - (3 - 2 + 6 + 8 - 3) + 8 - (9 - 3 + 8)$.
18. $3(4 - x) - 2(5 - 6x) = 8x + 4$.
19. $12 + (2a - 3c - 4b) - (3b - c - a - 8)$.
20. $5x - (3x - 2 + 2y + x) + 13y - (6 - 3x + 4)$.
21. Add $y - 20$, $4y + 6$, $2y + 4x - 13$, and $2x - 8y - 40$.
22. Subtract $16 - x + 2z - 4y$ from $3x - 5z - 8y$.
23. $19 + (2x - 7) - (31 - 4x - 8 - 2x) = 5x + 7$.
24. $16 + 5x - (8x + 9 - 4x + 17) = 8x - 3$.
25. $6x - 3 - (4x + 8 - 9x) - (5x - 2) = x + 11$.

MULTIPLICATION OF POLYNOMIALS

The following principle is useful in multiplying one monomial by another.

82. Principle XIII. *To obtain the product of two or more factors, these may be arranged and multiplied in any desired order.*

The truth of this principle is clear from examples such as:

$$2 \cdot 3 \cdot 5 = 2 \cdot 5 \cdot 3 = 5 \cdot 3 \cdot 2 = 5 \cdot (3 \cdot 2) = 2 \cdot (3 \cdot 5) = 30.$$

Historical Note. Principle XIII like Principle XI states two fundamental laws of algebra: (1) *the associative law of factors* first so called by F. S. Servois; (2) *the commutative law of factors*, first so called by Sir William Hamilton.

83. In multiplying algebraic expressions, the same factor frequently occurs more than once in the same term.

Thus we may have $5 \cdot 5$ or $a \cdot a$. These are written 5^2 and a^2 respectively, and read *5 square* and *a square*.

In these expressions the 2 is called an **exponent** and shows that the number above which it is written is to be used *twice as a factor*.

This is a convenient way of abbreviating written expressions.

E.g. $5a \cdot a = 5a^2$, $5a \cdot 3a = (5 \cdot 3) \cdot (a \cdot a) = 15a^2$.
 $7x \cdot 7x = 7 \cdot 7xx = 7^2x^2$, $ay \cdot ay = aayy = a^2y^2$.

84. The product of two binomials, such as $5 + 8$ and $5 + 3$, may be obtained in two ways:

$$(1) (5 + 3)(5 + 8) = 8 \cdot 13 = 104.$$

$$(2) (5 + 3)(5 + 8) = 5(5 + 8) + 3(5 + 8) = 5^2 + 5 \cdot 8 + 3 \cdot 5 + 3 \cdot 8 = 104.$$

	5	8
8	8 · 5	8 · 8
5	5 · 5	5 · 8
	5	8

The second method is illustrated by the accompanying figure in which $5 + 8$ is the length of a rectangle and $5 + 3$ is its width. The total area is the product $(5 + 3) \cdot (5 + 8)$ and is composed of the four small

areas, 5^2 , $5 \cdot 8$, $3 \cdot 5$, and $3 \cdot 8$.

The second method here used for multiplying $(5 + 3)(5 + 8)$ is the only one available when the terms of the binomials cannot be combined.

Thus

$$\begin{aligned}(x + 4)(x + 6) &= x(x + 6) + 4(x + 6) = x^2 + 6x + 4x + 4 \cdot 6 \\ &= x^2 + 10x + 24,\end{aligned}$$

$$\text{and } (a + b)(m + n) = a(m + n) + b(m + n) = am + an + bm + bn.$$

Hence, to multiply two binomials, *multiply each term of one by every term of the other and add the products.*

85. In a manner similar to that just illustrated we may multiply two trinomials.

E.g. The product of $a + b + c$ and $m + n + r$, in which the letters represent any positive numbers, may represent the area of a rectangle, divided into small rectangles as follows:

	m	n	r
a	am	an	ar
b	bm	bn	br
c	cm	cn	cr

Hence, the product is :

$(a + b + c)(m + n + r) = am + bm + cm + an + bn + cn + ar + br + cr$,
in which each term of one trinomial is multiplied by every term of the other
and the products are added.

Evidently the same process is applicable to the product of
two such polynomials each containing any number of terms.

EXERCISES AND PROBLEMS

Find the following products :

1. $(x+1)(x+2)$. 3. $(u+7)(u+4)$. 5. $(t+3)(t+7)$.
2. $(x+3)(x+5)$. 4. $(a+8)(a+3)$. 6. $(y+9)(y+2)$.
7. $(8+1)(8+7)$. 14. $(5s+1)(s+5)$.
8. $(s+5)(s+3)$. 15. $(x+7)(3x+4)$.
9. $(a+b)(c+d)$. 16. $(a+4)(3a+1)$.
10. $(x+4)(x+3)$. 17. $(3+x)(2+5x)$.
11. $(x+y+z)(a+b+c)$. 18. $(a+b)(3a+7b)$.
12. $(2x+3)(x+2)$. 19. $(x+y)(2x+3y)$.
13. $(5+x)(6+x)$. 20. $(7x+4)(x+8)$.

21. A rectangle is 7 feet longer than it is wide. If its length is increased by 3 feet and its width increased by 2 feet, its area is increased by 60 square feet. What are its dimensions?

22. A field is 10 rods longer than it is wide. If its length is increased by 10 rods and its width increased by 5 rods, the area is increased by 640 square rods. What are the dimensions of the field?

23. A farmer has a plan for a granary which is to be 12 feet longer than wide. He finds that if the length is increased 8 feet and the width increased 2 feet, the floor space will be increased by 160 square feet. What are the dimensions?

24. If the length of a rectangular flower bed is increased 3 feet and its width increased 1 foot, its area will be increased by 19 square feet. What are its present dimensions, if its length is 4 feet greater than its width?

86. Polynomials with negative terms. The polynomials multiplied in the foregoing exercises contain positive terms only. The same process is applicable to polynomials containing negative terms, as is seen in the following examples:

Ex. 1. Find the product of $(7-4)$ and $(3+5)$. This product, written out term by term, would give

$$\begin{aligned}[7 + (-4)](3 + 5) &= 7 \cdot 3 + 7 \cdot 5 + (-4) \cdot 3 + (-4) \cdot 5 \\ &= 21 + 35 - 12 - 20 = 24.\end{aligned}$$

Also $(7-4)(3+5) = 3 \cdot 8 = 24$.

Ex. 2. Multiply $7-4$ and $8-3$.

$$\begin{aligned}(7-4)(8-3) &= [7 + (-4)][8 + (-3)] \\ &= 7 \cdot 8 + 7 \cdot (-3) + (-4) \cdot 8 + (-4)(-3) \\ &= 56 - 21 - 32 + 12 = 15.\end{aligned}$$

Also $(7-4)(8-3) = 3 \cdot 5 = 15$.

Similarly, $(x+5)(x-2) = x^2 + 5x - 2x - 10 = x^2 + 3x - 10$,
and $(x-3)(x-5) = x^2 - 3x - 5x + 15 = x^2 - 8x + 15$.

EXERCISES

Perform the following indicated operations:

- | | |
|--------------------|------------------------|
| 1. $(x-5)(x-3)$. | 9. $(a-b)(c+d)$. |
| 2. $(x-3)(x+4)$. | 10. $(a-b)(c-d)$. |
| 3. $(a-6)(a-1)$. | 11. $(x-4)(x-5)$. |
| 4. $(u+5)(u-3)$. | 12. $(a+b-c)(m-n)$. |
| 5. $(b+2)(b-7)$. | 13. $(a-b)(7a+3b)$. |
| 6. $(3-b)(4+b)$. | 14. $(5-y)(5x+3y)$. |
| 7. $(3+x)(7-3x)$. | 15. $(2a-3b+c)(m+n)$. |
| 8. $(n-4)(3-n)$. | 16. $(v-t)(7v-5t)$. |

The preceding exercises illustrate

Principle XIV

87. Rule. *The product of two polynomials is found by multiplying each term of one by every term of the other, and adding these products.*

88. If there are similar terms in either polynomial, these should be added first, thus putting each polynomial in as simple form as possible.

$$\begin{aligned} \text{E.g. } (3x + 2 - 2x)(4x + 3 - 3x) &= (x + 2)(x + 3) \\ &= x^2 + 2x + 3x + 6 = x^2 + 5x + 6. \end{aligned}$$

It should be observed that Principle XIV involves a repeated application of Principle II. Thus

$$(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd.$$

To simplify the process of combining similar terms in the product, it may be found convenient to arrange the work as in the following examples:

Ex. 1. Multiply $3x - 2$ by $2x - 5$.

$$\begin{array}{r} \text{Solution.} \qquad 3x - 2 \\ \qquad \qquad \quad 2x - 5 \\ \hline \qquad \qquad 6x^2 - 4x \\ \qquad \qquad - 15x + 10 \\ \hline \qquad \qquad 6x^2 - 19x + 10 \end{array}$$

Ex. 2. Multiply $3x - 2y + 2$ by $4x - 3y - 2$.

$$\begin{array}{r} \text{Solution.} \qquad 3x - 2y + 2 \\ \qquad \qquad \quad 4x - 3y - 2 \\ \hline \qquad \qquad 12x^2 - 8xy + 8x \\ \qquad \qquad \quad - 9xy \qquad + 6y^2 - 6y \\ \qquad \qquad \qquad \quad - 6x \qquad + 4y - 4 \\ \hline \qquad \qquad 12x^2 - 17xy + 2x + 6y^2 - 2y - 4 \end{array}$$

EXERCISES AND PROBLEMS

Perform the following indicated operations:

In each case simplify the expressions within the parentheses as much as possible before multiplying:

1. $(x-7)(3x+4)$. 2. $(x-2)(9x+4)$. 3. $(a-x)(9x+4a)$.
4. $(5x+3y-4x-2y)(6y+3x-2y+y)$.

5. $(13a - b - 12a)(2b - 3a)$.
6. $(xy - 5xy + 4)(8y - 3 - 7y)$.
7. $(11b + 3a)(2b - 3b + 5)$.
8. $(6 - 4x + 3x)(7x + y - 3x + 1)$.
9. $(x - y + 3)(5x - 3y + 5)$. 10. $(a - 13n)(a - n + 8)$.
11. $(x - 2 + y)(4y - 3x)$.
12. $(11b - a - 10b)(6a - 3b - 2a)$.
13. $(7 + y - x)(2y + x - 1)$. — 14. $(5x + 3y - 1)(x - 2)$.
15. $(-7a - 1 + 8a)(5a - 8 - 3a)$.

Solve the following equations and check the results:

16. $(x + 2)(x + 3) = (x - 3)(x + 10) + 10$.
17. $(5x - 4)(6 - x) - 97 = (x - 1)(6 - 5x)$.
18. $(3n - 1)(18 - n) = (n + 6)(16 - 3n)$.
19. $(7 - a)(9a - 8) = 31 + (36 - 9a)(a + 2)$.
20. $(4a + 4)(a - 3) = (4a + 1)(a + 7) - 13a + 221$.
21. $(n + 6)(3n - 4) - 14 = (n + 8)(3n - 3)$.
22. $(8n + 6)(10 - n) + 150 = (1 - n)(8n + 3)$.
23. $(a - 1)(13 - 6a) = (6a - 3)(8 - a) - 21$.
24. $(7x - 13)(6 - x) - (x + 4)(3 - 7x) = 70$.
25. Find two numbers whose difference is 6 and whose product is 180 greater than the square of the smaller.
26. There are four consecutive *even* integers such that the product of the first and second is 40 less than the product of the third and fourth. What are the numbers?
27. There are four consecutive integers such that the product of the first and third is 223 less than the product of the second and fourth. What are the numbers?
28. Find four numbers such that the second is 5 greater than the first, the third 5 greater than the second, and the fourth 5 greater than the third. The product of the first and second is 250 less than the product of the third and fourth.

29. A club makes an equal assessment on its members each year to raise a certain fixed sum. One year each member pays a number of dollars equal to the number of members of the club less 175. The following year, when the club has 50 more members, each member pays \$5 less than the preceding year. What was the membership of the club the first year and how much did each pay?

PROBLEMS ON RECTANGLES AND TRIANGLES

30. A rectangle is 10 inches longer than wide. Express its area in terms of the width w . If the width is increased by 4 and the length by 6 inches, express the area in terms of w .

31. A rectangle is 8 inches longer than wide. Express its area in terms of the width w after the width is increased 4 inches and the length decreased 10 inches.

32. A rectangle is 5 feet longer than it is wide. If it were 3 feet wider and 2 feet shorter, it would contain 15 square feet more. Find the dimensions of the rectangle.

33. A rectangle is 6 feet longer and 4 feet narrower than a square of equal area. Find the side of the square and the sides of the rectangle.

If b is the base of a triangle, h its altitude (height), and a its area, then $area = \frac{1}{2}(base \times altitude)$;

i.e.
$$a = \frac{bh}{2}.$$

34. The base of a triangle is 2 inches less than its altitude a . Express the area in terms of a .

35. The altitude of a triangle is 7 greater than its base b . If the altitude is decreased by 8 and the base by 6, express its area in terms of b .

36. The altitude of a triangle is 16 inches less than the base. If the altitude is increased by 3 inches and the base by 2 inches, the area is increased by 52 square inches. Find the base and altitude of the triangle.

SQUARES OF BINOMIALS

89. Just as x^2 is written instead of $x \cdot x$, so $(a+b)^2$ is written instead of $(a+b)(a+b)$. The square of a binomial is found by multiplying the binomial by itself as in § 84.

$$\text{E.g. } (a+b)(a+b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

Hence,

$$(a+b)^2 = a^2 + 2ab + b^2. \quad \text{I}$$

	a	b
b	ba	b^2
a	a^2	ab

This product is illustrated in the accompanying figure, and is evidently a special case of the type exhibited in the figures, page 68.

Translated into words this identity is: *The square of the sum of any two numbers is equal to the square of the first plus twice the product of the two numbers plus the square of the second.*

By formula I we may square any binomial sum.

$$\text{E.g. } (3x+2y)^2 = (3x)^2 + 2 \cdot (3x)(2y) + (2y)^2 = 9x^2 + 12xy + 4y^2.$$

EXERCISES

Find the following products. Read the first four at sight.

- | | | |
|-----------------|------------------|-------------------|
| 1. $(a+2)^2$. | 6. $(1+3a)^2$. | 11. $(5x+4)^2$. |
| 2. $(x+3)^2$. | 7. $(2a+3b)^2$. | 12. $(3a+7c)^2$. |
| 3. $(m+n)^2$. | 8. $(2x+1)^2$. | 13. $(1+8a)^2$. |
| 4. $(5+b)^2$. | 9. $(3y+2)^2$. | 14. $(x+11y)^2$. |
| 5. $(2a+b)^2$. | 10. $(c+4b)^2$. | 15. $(a+9b)^2$. |

90. Similarly, we obtain the square of the difference of two numbers:

$$(a-b)^2 = a^2 - 2ab + b^2. \quad \text{II}$$

Translate this identity into words.

Ex. By use of formula II, find the square of $a-3b$.

$$\text{Solution. } (a-3b)^2 = a^2 + 2 \cdot a(-3b) + (-3b)^2 = a^2 - 6ab + 9b^2.$$

While these squares are ordinary products of binomials and may be found by Principle XIV, they are of special importance and should be studied until they can be given from memory at any time.

EXERCISES

Find the following products. Read the first five at sight:

- | | | |
|----------------|------------------|-------------------|
| 1. $(a-3)^2$. | 6. $(1-2b)^2$. | 11. $(2x-3y)^2$. |
| 2. $(b-4)^2$. | 7. $(3c-1)^2$. | 12. $(7c-3a)^2$. |
| 3. $(c-d)^2$. | 8. $(2y-3)^2$. | 13. $(x-4y)^2$. |
| 4. $(x-7)^2$. | 9. $(3a-2b)^2$. | 14. $(1-4z)^2$. |
| 5. $(3-n)^2$. | 10. $(c-3b)^2$. | 15. $(a-7b)^2$. |

There are two other special products which should be memorized; namely, III and IV below.

91. **Examples.** Find the products,

$$(x+2)(x-2), (x+5)(x-5), (x+9)(x-9).$$

In each of these *one factor is the sum of two numbers and the other is the difference between the same numbers, while the product is the difference of the squares of these numbers.*

This is expressed by the formula

$$(x+a)(x-a) = x^2 - a^2. \quad \text{III}$$

By means of this formula the product of the sum and difference of any two number expressions may be found.

$$E.g. \quad (3a+2b)(3a-2b) = (3a)^2 - (2b)^2 = 9a^2 - 4b^2.$$

Verify this by performing the multiplication.

EXERCISES

Read the following products:

- | | |
|-----------------------|------------------------|
| 1. $(a+1)(a-1)$. | 9. $(1-7y)(1+7y)$. |
| 2. $(a+3)(a-3)$. | 10. $(a-4b)(a+4b)$. |
| 3. $(k-b)(k+b)$. | 11. $(6a-3b)(6a+3b)$. |
| 4. $(3-x)(3+x)$. | 12. $(7-9a)(7+9a)$. |
| 5. $(2a+3b)(2a-3b)$. | 13. $(2c+1)(2c-1)$. |
| 6. $(a+2b)(a-2b)$. | 14. $(3a+b)(3a-b)$. |
| 7. $(2b-1)(2b+1)$. | 15. $(5k+3h)(5k-3h)$. |
| 8. $(1+3x)(1-3x)$. | 16. $(9m+3n)(9m-3n)$. |

92. **Examples.** Find the products :

$$(x+2)(x+3) \qquad (x+4)(x-7)$$

$$(x+5)(x-2) \qquad (x-5)(x-3)$$

Each of these products when simplified consists of three terms, of which the first is x^2 , the last is the product of the second terms of the factors, and the coefficient of x in the middle term is the algebraic sum of the second terms of the factors.

This is expressed by the formula :

$$(x+a)(x+b) = x^2 + (a+b)x + ab. \qquad \text{IV}$$

Verify this by performing the multiplication.

EXERCISES

✓ Write the following products. Also try to read them at sight.

1. $(x+7)(x+3).$

6. $(a-8)(a+10).$

2. $(x+9)(x+6).$

7. $(a+7)(a+6).$

3. $(y+6)(y-2).$

8. $(a-7)(a+6).$

4. $(y-8)(y+3).$

9. $(ab+3)(ab+7).$

5. $(c-4)(c-2).$

10. $(ab-5)(ab-3).$

In the formula $(x+a)(x+b) = x^2 + (a+b)x + ab$, replace a and b by the following values and simplify the results :

11. $a=5, b=3.$

13. $a=6, b=-11.$

12. $a=8, b=-7.$

14. $a=-5, b=-7.$

15. Find the square of 42 by writing it as a binomial, $40+2.$

16. Square the following numbers by writing each as a binomial sum: 51, 53, 93, 91, 102, 202, 301.

17. Find the square of 29 by writing it as a binomial, $30-1.$

18. Square the following numbers by first writing each as a binomial difference: 28, 38, 89, 77, 99, 198, 499, 998, 999.

19. Find the product of 41 and 39, first indicating the product thus, $(40+1)(40-1).$

20. Find the following products by writing each pair of factors as the sum and difference of two numbers:

(1) $62 \cdot 58$.

(3) $53 \cdot 47$.

(5) $17 \cdot 13$.

(2) $27 \cdot 33$.

(4) $102 \cdot 98$.

(6) $99 \cdot 101$.

EQUATIONS AND PROBLEMS

Solve the following equations, verifying except where the answer is given.

1. $(a+4)^2 + (a-1)(2a+5) = (a+4)(3a+2)$.

2. $(a-1)(3a-1) - (a+1)^2 = 2a^2 - 18$.

3. $(6-a)^2 + (a-3)(2a-5) = (3a+1)(a-3) + 84$.

4. $(7a-18)(a+4) - (a-1)^2 = 6(a+2)^2 - 79$.

5. $(2b-30)(b-1) - 5b^2 = 6b - 3(b+5)^2 + 65$.

6. $(5-c)^2 + (7-c)^2 + (9-c)^2 = (c-1)(3c-58) - 93$.

7. $(5c-3)(2+c) - 4(c-1)^2 = (c+1)^2 + 54$.

8. $(8-4c)(5-c) = (c+1)^2 + (c+3)(3c-8) + 218$.

9. $(y-1)^2 + 4(y+1)^2 + (1-y)(5y+6) = 15y - 29$.

10. $x(x+3) + (x+1)(x+2) = 2x(x+5) + 2$.

11. $x^2 = (x-3)(x+6) - 12$.

12. $(5+5x)(3-x) + 2(x+1)^2 + 3(x+1)(x-7) = 17(x+1)$.

13. $(8+3x)(4-x) + (x-1)(x-2) + 2(x+5)^2 = 105$.

✓ 14. $(5-b)(6b+5) + 4(b-3)^2 = 20 - 2(b+1)^2 + 3 + 16b$.

Answer $3\frac{1}{11}$.

✓ 15. There is a square field such that if its dimensions are increased by 5 rods, its area is increased 625 square rods. How large is the field?

Suggestion: If a side of the original field is w , then its area is w^2 , and the area of the enlarged field is $(w+5)^2$.

✓ 16. A rectangle is 9 feet longer than it is wide. A square whose side is 3 feet longer than the width of the rectangle is equal to the rectangle in area. What are the dimensions of the rectangle?

17. A boy has a certain number of pennies which he attempts to arrange in a solid square. With a certain number on each side of the square he has 10 left over. Making each side of the square one larger, he lacks 7 of completing it. How many pennies has he?

18. A room is 7 feet longer than it is wide. A square room whose side is 3 feet greater than the width of the first room is equal to it in area. What are the dimensions of the first room?

19. Find two consecutive integers whose squares differ by 51.

20. Find two consecutive integers whose squares differ by 97.

21. Find two consecutive integers whose squares differ by a . Show from the form of the equation obtained that a must be an *odd* integer.

22. There are four consecutive integers such that the sum of the squares of the last two exceeds the sum of the squares of the first two by 20. What are the numbers?

23. Two square pieces of land require together 360 rods of fence? If the difference in the area of the pieces is 900 square rods, how large is each piece? (*Hint*: $x^2 - (90 - x)^2 = 900$.)

24. There is a square such that if one side is increased by 12 feet and the other side decreased by 8 feet the resulting rectangle will have the same area as the square. Find the side of the square.

25. A regiment was drawn up in a solid square. After 50 men had been removed the officer attempted to draw up the square by putting one man less on each side, when he found he had 9 men left over. How many men in the regiment?

26. There is a rectangle whose length exceeds its width by 11 rods. A square whose side is 5 rods greater than the width of the rectangle is equal to it in area. What are the dimensions of the rectangle?

REVIEW QUESTIONS

1. What is a polynomial? a term? How are polynomials classified? What are similar terms? By what principle are similar terms added? By what principle are they subtracted?

2. In adding or subtracting polynomials how may the terms be arranged for convenience? State the principle on which this is based?

3. What is the principle for removing a parenthesis when preceded by the sign +? By the sign -? How may Principle XII be used for inclosing terms within a parenthesis?

4. In finding the product of two or more numbers how may the factors be arranged for convenience? State the principle on which this is based.

5. Make a diagram to show how to multiply $(7 + 4)$ by $(11 + 8)$ without first uniting the terms of the binomials. Multiply $(a + b)$ by $(c + d)$ in the same manner.

Multiply $(12 - 3)$ by $(9 - 7)$ in two ways and compare results. State the principle by which two polynomials are multiplied.

6. Describe a convenient manner of arranging the work in multiplying polynomials. What kind of terms in the product are placed in the same column? Find the product of $7x - 3y + 1$ and $2x - 8y - 3$, arranging your work this way.

7. State in words what is the square of the binomial $x + a$; of the binomial $x - a$.

8. Translate into words the formula $(x + a)(x - a) = x^2 - a^2$.

9. Translate into words the formula

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

Interpret this formula for various positive and negative values of a and b .

10. Complete your list of Principles stated in symbols up to and including Principle XIV.

DRILL EXERCISES

1. $(x-1)(2x-2)+(x-5)^2=(3-x)(24-3x)-7$.
2. From $3-4a-5c+8x^2$ subtract $2x^2-2a-4c+8$.
3. $(4ab-6ac-5ad)(b-c+d)$.
4. $(17x+3)(x-1)+8=(2-x)(6-17x)+19$.
5. $5-(a+b-c-d+8)+(3+a+c-d)-5$.
6. Add $6a+9$, $8a-13$, $46a-8$, and $6-54a$.
7. $(a-2)(6a-4)+2(a-1)^2=(6-a)(30-8a)+4$.
8. From $6(a+2)+3(c+4)-2(b-d)$
subtract $2(a+2)-2(c+4)+3(b-d)$.
9. Add $12a^2b^2c+8ax$, $6ax-8a^2b^2c$, and $2ax+3a^2b^2c$.
10. Add $5xy^2+3x^2y+4xy$, $2x^2y-6xy^2-3xy$, and $4xy$.
11. Add $6ab-3c-2a$, $2c-4ab-5a$, $5c-a+ab$, and
 $3+5a-2c-3ab$.
12. $(n-4)(6-3n)-(6-n)^2-10=-4n(n-4)$.
13. From $35ab-8x-9z+13$ subtract $16ab-4z+5x+8$.
14. $(n+2)^2+(n-1)^2+(n+1)^2=3n(n+2)+60n+130$.
15. Subtract $5a-8x-6y$ from $13x+14y-15z-4a$.
16. From $9y-4x-6z-3b$ subtract $8-9y-3x-2z$.
17. $2x+4-6(5x-8-7x)+2-4x=6(2-3x)-42$.
18. $-(7+4x-8-2x)+4-2x=6x+25$.
19. $(a-1+b-c-d)(4a+5b+3c-2d)$.
20. $(4ax-3ay+5az-8)(x+y-z+2)$.
21. $(3a-2b+4c)(2a+3b-c)$.
22. $7-(3a-2b-4a)+b+2a-(3b-2a-a)$.
23. $8x+(5y-5)+(2y-1)-(13y+8x-17)$.

CHAPTER V

SIMPLE FRACTIONS

93. Definitions. In arithmetic a fraction such as $\frac{2}{3}$ is usually regarded as 2 of the 3 equal parts of a unit.

However, a fraction such as $\frac{5}{3\frac{1}{2}}$ cannot be regarded in this way, since a unit cannot be divided into $3\frac{1}{2}$ equal parts. $\frac{5}{3\frac{1}{2}}$ indicates that 5 is to be divided by $3\frac{1}{2}$; i.e. $\frac{5}{3\frac{1}{2}} = 5 \div 3\frac{1}{2}$.

In algebra any fraction is usually regarded as an **indicated division** in which the numerator is the dividend and the denominator is the divisor.

Thus, $\frac{a}{b}$ is understood to mean $a \div b$.

The numerator and denominator are together called the **terms of the fraction**.

Operations on algebraic fractions are performed in accordance with the same rules as in arithmetic.

EXERCISES

Supply the missing numerator in each of the following:

1. $\frac{1}{2} = \frac{\quad}{4}$

4. $\frac{a}{b} = \frac{\quad}{cb}$

7. $\frac{1}{a-1} = \frac{\quad}{1-a}$

2. $\frac{3}{4} = \frac{\quad}{12}$

5. $\frac{m}{n} = \frac{\quad}{3n}$

8. $\frac{1}{a+1} = \frac{\quad}{(a+1)(a-1)}$

3. $\frac{1}{a} = \frac{\quad}{ka}$

6. $\frac{1}{a} = \frac{\quad}{-2a}$

9. $\frac{1}{b+2} = \frac{\quad}{(b+3)(b+2)}$

The preceding examples illustrate

Principle XV

94. Rule. *Both terms of a fraction may be multiplied or divided by the same number without changing its value.*

95. Definition. The lowest common multiple (L. C. M.) of two or more numbers is the least number which contains as factors all the factors of these numbers.

E.g. 12 is the L. C. M. of 4 and 6. abc is the L. C. M. of ab , bc , and ac . $(a+b)(a+1)$ is the L. C. M. of $(a+b)$ and $(a+1)$.

EXERCISES

Find the L. C. M. of each of the following:

- | | | |
|---------------|------------------|------------------|
| 1. 3, 5. | 5. $a, bk.$ | 9. $b+2, b+3.$ |
| 2. 3, 4, 6. | 6. $ab, bc, ac.$ | 10. $1-a, 1-2a.$ |
| 3. 6, 48, 24. | 7. $3a, 2b, 4c.$ | 11. $2x+3, x-4.$ |
| 4. $a, b.$ | 8. $a+1, a-1.$ | 12. $m+3, m-5.$ |

REDUCTION TO A COMMON DENOMINATOR

96. Examples. 1. Reduce $\frac{1}{2}$ and $\frac{1}{3}$ to a common denominator. How is the common denominator related to the denominators 2 and 3?

2. Reduce $\frac{1}{a}$ and $\frac{1}{b}$ to a common denominator. How is the common denominator related to the denominators a and b ?

3. Reduce $\frac{2}{a+1}$, $\frac{3}{(a+1)(a+2)}$, $\frac{4}{a+2}$ to a common denominator.

Solution. The required denominator is the L. C. M. of the given denominators; that is, $(a+1)(a+2)$.

Hence, $\frac{2}{a+1} = \frac{2(a+2)}{(a+1)(a+2)}$ and $\frac{4}{a+2} = \frac{4(a+1)}{(a+1)(a+2)}$.
 $\frac{3}{(a+1)(a+2)}$ already has the required denominator.

EXERCISES

Reduce each of the following to a common denominator:

1. $\frac{1}{a}, \frac{1}{b}$.
4. $\frac{1}{ab}, \frac{1}{ac}$.
7. $\frac{1}{ab}, \frac{1}{bc}, \frac{1}{ac}$.
2. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.
5. $\frac{a}{bc}, \frac{b}{ac}$.
8. $\frac{a}{bc}, \frac{b}{ac}, \frac{c}{ab}$.
3. $\frac{1}{a}, \frac{1}{-a}$.
6. $\frac{a}{xy}, \frac{b}{xz}$.
9. $\frac{m}{xy}, \frac{n}{xz}, \frac{s}{yz}$.
10. $\frac{1}{x+1}, \frac{1}{x+2}$.
13. $\frac{1}{1-k}, \frac{1}{k-1}$.
16. $\frac{3}{n-4}, \frac{2}{(n-1)(n-4)}$.
11. $\frac{2}{a-3}, \frac{1}{a+4}$.
14. $\frac{2}{x+2}, \frac{3}{x-2}$.
17. $\frac{1}{x-1}, \frac{1}{(x-1)(x-2)}$.
12. $\frac{a}{a-b}, \frac{c}{c-d}$.
15. $\frac{a}{a-b}, \frac{b}{a+b}$.
18. $\frac{2}{a-b}, \frac{3}{(a-b)(c-d)}$.

ADDITION AND SUBTRACTION

97. **Examples.** 1. Add $\frac{3}{4}$ and $\frac{5}{8}$. How is the numerator of the sum found after the fractions have been reduced to a common denominator.

2. Add $\frac{1}{a+1}$ and $\frac{2}{a+3}$.

Solution. $\frac{1}{a+1} = \frac{a+3}{(a+1)(a+3)}; \frac{2}{a+3} = \frac{2(a+1)}{(a+1)(a+3)}$.

Hence, $\frac{1}{a+1} + \frac{2}{a+3} = \frac{a+3}{(a+1)(a+3)} + \frac{2(a+1)}{(a+1)(a+3)}$
 $= \frac{a+3+2(a+1)}{(a+1)(a+3)} = \frac{3a+5}{(a+1)(a+3)}$.

EXERCISES

Perform the following additions:

$$1. \frac{1}{a} + \frac{1}{b}.$$

$$5. \frac{2}{a} + \frac{1}{-a}.$$

$$2. \frac{1}{ab} + \frac{1}{ac}.$$

$$6. \frac{1}{1-k} + \frac{1}{k-1}.$$

$$3. \frac{1}{a+1} + \frac{1}{a-1}.$$

$$7. \frac{3}{n-4} + \frac{2}{(n-1)(n-4)}.$$

$$4. \frac{2}{x+1} + \frac{3}{x+2}.$$

$$8. \frac{2}{x+2} + \frac{3}{x-2}.$$

98. **Examples.** 1. From $\frac{5}{7}$ subtract $\frac{2}{3}$. How is the numerator of the difference found after the fractions have been reduced to a common denominator?

2. From $\frac{3}{x-4}$ subtract $\frac{2}{x+3}$.

Solution:

$$\begin{aligned} \frac{3}{x-4} - \frac{2}{x+3} &= \frac{3(x+3)}{(x-4)(x+3)} - \frac{2(x-4)}{(x-4)(x+3)} \\ &= \frac{3(x+3) - 2(x-4)}{(x-4)(x+3)} = \frac{x+17}{(x-4)(x+3)}. \end{aligned}$$

EXERCISES

Perform the following subtractions:

$$1. \frac{1}{a} - \frac{1}{b}.$$

$$5. \frac{2}{-a} - \frac{1}{a}.$$

$$2. \frac{1}{ab} - \frac{1}{ac}.$$

$$6. \frac{1}{1-k} - \frac{1}{k-1}.$$

$$3. \frac{1}{a+1} - \frac{1}{a-1}.$$

$$7. \frac{2}{x+2} - \frac{3}{x-2}.$$

$$4. \frac{2}{x+1} - \frac{3}{x+2}.$$

$$8. \frac{3}{n-4} - \frac{2}{(n-1)(n-4)}.$$

EXERCISES

Perform the following additions and subtractions:

1. $\frac{1}{x-2} - \frac{1}{x+1}$.

10. $\frac{3}{a+b} - \frac{2}{a-b} + \frac{1}{a}$.

2. $\frac{1}{a+4} - \frac{2}{a-3}$.

11. $\frac{a}{1-a} + \frac{1}{a-1} + \frac{1}{a}$.

3. $\frac{1}{x-1} - \frac{1}{1-x}$.

12. $\frac{3}{x-y} - \frac{1}{y} + \frac{1}{x}$.

4. $\frac{1}{x} - \frac{1}{y}$.

13. $\frac{x}{x-y} + \frac{x}{y} - \frac{1}{y}$.

5. $\frac{a}{a-b} - \frac{b}{a+b}$.

14. $\frac{1}{x} + \frac{a}{y} - \frac{b}{z}$.

6. $\frac{3}{k-1} - \frac{2}{k+1}$.

15. $\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$.

7. $\frac{4}{c-2} - \frac{3}{(c-1)(c-2)}$.

16. $\frac{a}{(x-1)(x+3)} - \frac{b}{x+3}$.

8. $\frac{1}{a} - \frac{1}{b} + \frac{1}{c}$.

17. $\frac{1}{x+y} + \frac{1}{x-y} + \frac{1}{(x+y)(x-y)}$.

9. $\frac{1}{ab} + \frac{1}{bc} - \frac{1}{abc}$.

18. $\frac{1}{a+1} + \frac{1}{a-1} - \frac{1}{(a+1)(a-1)}$.

MULTIPLICATION AND DIVISION

99. Examples. 1. Multiply $\frac{3}{4}$ by 5, also $\frac{a}{b}$ by c .

2. Multiply $\frac{5}{8}$ by 4, also $\frac{a}{bc}$ by c .

3. Multiply $\frac{3}{8}$ by $\frac{5}{7}$, also $\frac{a}{b}$ by $\frac{c}{d}$.

4. Multiply $\frac{4(x+3)}{(x-1)(x+2)}$ by $\frac{3(x-1)}{4(x+3)(x-2)}$.

Solution. $\frac{\cancel{4}(x+3)}{(\cancel{x-1})(x+2)} \times \frac{3(\cancel{x-1})}{\cancel{4}(\cancel{x+3})(x-2)} = \frac{3}{(x+2)(x-2)} = \frac{3}{x^2-4}$.

The first step is to cancel all possible factors.

EXERCISES

Perform the following multiplications:

1. $\frac{ab}{c} \times \frac{c}{a}$.

3. $\frac{abc}{xy} \times \frac{bx}{ac}$.

5. $\frac{mxy}{nyc} \times \frac{ny}{mx}$.

2. $\frac{ab}{cx} \times \frac{c^2x^2}{a}$.

4. $\frac{cx^2}{by^2} \times \frac{cy}{dx}$.

6. $\frac{rst}{xy} \times \frac{xy^2}{r^2s}$.

7. $\frac{3}{n+8} \times (n+1)$.

9. $\frac{3c}{(3-c)(3+c)} \times \frac{(c-3)(c+3)}{6cd}$.

8. $\frac{2}{(a-2)(a+3)} \times 3(a-2)$.

10. $\frac{3a+b}{2+3x} \times \frac{(3x+2)(2a-b)}{(a+b)(3a+b)}$.

11. $\frac{5ab}{2cd} \times \frac{4bc}{10a}$.

12. $\frac{(a+2b)(a+2b)}{a-2b} \times \frac{(2a-b)(a-2b)}{(3a+b)(a+2b)}$.

100. Examples. 1. Divide $\frac{1}{2}$ by 2, also $\frac{ab}{c}$ by a .

2. Divide $\frac{2}{3}$ by 7, also $\frac{a}{b}$ by c .

3. Divide $\frac{3}{4}$ by $\frac{5}{7}$, also $\frac{a}{b}$ by $\frac{c}{d}$.

4. Divide $\frac{ab}{(x-1)(x-2)}$ by $\frac{b}{x-1}$.

Solution. $\frac{ab}{(x-1)(x-2)} \div \frac{b}{x-1} = \frac{ab}{(x-1)(x-2)} \times \frac{x-1}{b} = \frac{a}{x-2}$.

After inverting the terms of the divisor cancel all possible factors.

EXERCISES

Perform the following divisions:

1. $\frac{ab}{c} \div c$.

3. $\frac{abc}{xy} \div \frac{bc}{ax}$.

5. $\frac{mxy}{ny} \div \frac{ny}{mx}$.

2. $\frac{ab}{c} \div \frac{bx}{c}$.

4. $\frac{ax^2}{by^2} \div \frac{ax}{cy}$.

6. $\frac{rst}{xy} \div \frac{r^2s}{x^2y}$.

7. $\frac{abc}{x-1} \div ab$. 9. $\frac{a+1}{a-2} \div \frac{3(a+1)}{a+4}$. 11. $\frac{4k^2}{-9abc^2} \div -\frac{2k}{ab}$.
8. $\frac{xy}{a-b} \div (a+b)$. 10. $\frac{24ab}{25c} \div \frac{16ab}{15}$. 12. $\frac{ab^2}{(x+y)^2} \div \frac{ab}{x+y}$.
13. $\frac{6a-3}{5x} \div \frac{2a-3}{15x^2}$. 14. $\frac{(2+x)(2-x)}{3x(x+1)} \div \frac{(2-x)^2}{x(x-2)}$.

NOTE.—A full treatment of fractions will be found toward the end of this course. To give such a treatment here would necessitate a complete study of factoring and consequently the postponement of much important work on the solution of equations and problems which naturally belongs early in the course. The treatment given in this chapter is based directly on *Arithmetic* and is sufficient for the solution of all problems depending on simple equations which naturally occur in an elementary course.

EQUATIONS INVOLVING FRACTIONS

$$\text{Solve } n + \frac{n}{2} + \frac{n}{3} = 88. \quad (1)$$

First Solution. The coefficients of n are 1 , $\frac{1}{2}$, and $\frac{1}{3}$.

$$\text{Applying Principle I, } (1 + \frac{1}{2} + \frac{1}{3})n = 1\frac{5}{6}n = 88. \quad (2)$$

$$\text{By } D \mid 1\frac{5}{6}, \quad n = 88 \div 1\frac{5}{6} = 48. \quad (3)$$

Second Solution. Multiply both members of (1) by 6.

$$\text{That is, by } M \mid 6, \quad 6n + \frac{6n}{2} + \frac{6n}{3} = 528. \quad (2)$$

$$\text{By } F, V, \quad 6n + 3n + 2n = 528. \quad (3)$$

$$\text{By } F, I, \quad 11n = 528. \quad (4)$$

$$\text{By } D \mid 11, \quad n = 48, \text{ as before.} \quad (5)$$

The object is to multiply both members of the equation by such a number as will cancel each denominator. Hence the multiplier must contain each denominator as a factor.

Evidently 12 or 18 might have been chosen for this purpose, but not 8 or 10. 6 is the smallest number which will cancel both 2 and 3.

101. The process explained in the second solution above is called **clearing of fractions**.

As another illustration solve the equation

$$\frac{4}{x+1} - \frac{1}{x-1} = \frac{4}{(x+1)(x-1)}. \quad (1)$$

Here the lowest multiple available is $(x+1)(x-1)$.

$$\text{Hence, } \frac{4\cancel{(x+1)}(x-1)}{\cancel{x+1}} - \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \frac{4\cancel{(x+1)}\cancel{(x-1)}}{\cancel{(x+1)}\cancel{(x-1)}}. \quad (2)$$

By F, V, each denominator is now cancelled,

$$4(x-1) - (x+1) = 4. \quad (3)$$

$$\text{By F, II, XII, } 4x - 4 - x - 1 = 4. \quad (4)$$

$$\text{By F, I and } A \mid 5, \quad 3x = 9. \quad (5)$$

$$\text{By } D \mid 3, \quad x = 3. \quad (6)$$

Check. Substitute $x = 3$ in (1) and get

$$1 - \frac{1}{2} = \frac{1}{2} \text{ or } \frac{1}{2} = \frac{1}{2}.$$

After a little practice step (2) should be performed mentally and equation (3) should be derived immediately from (1).

EXERCISES AND PROBLEMS

Solve the following equations, indicating the principles used at each step. Check each solution by substituting the value obtained in the original equation. Translate the first four into problems.

For instance, from the equation in Ex. 2: Find a number such that when increased by its half, its third, and its fourth, the sum is 25.

$$1. \quad \frac{n}{2} + \frac{n}{3} = 5. \quad 3. \quad \frac{n}{4} + 4n - \frac{5n}{3} = \frac{3n}{2} + 26.$$

$$2. \quad n + \frac{n}{2} + \frac{n}{3} + \frac{n}{4} = 25. \quad 4. \quad \frac{n}{2} + \frac{n}{3} - \frac{n}{4} + \frac{n}{10} = 82.$$

$$5. \quad 7x + \frac{4x}{7} + \frac{3x}{5} + 23 = \frac{x}{5} + \frac{4x}{10} + 5x + 113$$

$$6. \quad 4n + \frac{6n}{7} = \frac{3n+2}{2} + 46.$$

$$7. 12 + \frac{4(9x+6)}{3} - \frac{2(3+11x)}{5} = \frac{5(4x+4)}{3} + 14.$$

$$8. 13b + 7 - \frac{9b+8}{7} = \frac{2b+9}{5} + 38.$$

$$9. \frac{5a+7}{2} + \frac{2a+4}{3} = \frac{3a+9}{4} + 5.$$

$$10. \frac{17a-5}{3} - \frac{10a+2}{4} = \frac{5a+7}{2} - 5. \quad 12. \frac{3}{x} - \frac{1}{2} = \frac{1}{x}.$$

$$11. \frac{5x+3}{2} + \frac{5(2x+10)}{5} = 2x + 24. \quad 13. \frac{1}{3x} + \frac{1}{2x} = \frac{1}{6}.$$

$$14. \frac{1}{4x} + \frac{1}{3x} = \frac{5}{6x} + \frac{1}{2}. \quad 15. \frac{1}{x} + \frac{2}{x-1} = \frac{8}{x(x-1)}.$$

$$16. \frac{4}{x+1} - \frac{3}{x-1} = \frac{-2}{(x+1)(x-1)}.$$

$$17. \frac{1}{x-1} + \frac{1}{x+1} = \frac{4}{(x-1)(x+1)}.$$

18. The sum of two numbers is 12, and the first number is $\frac{1}{3}$ as great as the second. What are the numbers?

19. The smaller of two numbers is $\frac{2}{3}$ of the larger. If their sum is 66, what are the numbers?

20. Find two consecutive integers such that $\frac{5}{7}$ of the first minus $\frac{3}{11}$ of the second equals 9.

21. Find three consecutive integers such that $\frac{1}{3}$ of the first plus the second minus $\frac{1}{2}$ the third equals 5.

22. Find three consecutive integers such that $\frac{1}{2}$ of the first plus $\frac{1}{3}$ of the second minus $\frac{1}{10}$ of the third equals 28.

23. There are three numbers such that the second is 4 more than 9 times the first, and the third is 2 more than 6 times the first. If $\frac{1}{3}$ of the third is subtracted from $\frac{1}{2}$ of the second, the remainder is 3. Find the numbers.

24. There are three numbers such that the second is 2 more than 9 times the first and the third is 5 more than 11 times the first. The remainder when $\frac{1}{3}$ of the third is subtracted from $\frac{1}{2}$ of the second is one. Find the numbers.

- 25. What number must be subtracted from both the numerator and the denominator of the fraction $\frac{5}{8}$ in order to make the result equal to $\frac{1}{2}$.

X - 26. What number must be subtracted from both numerator and denominator of the fraction $\frac{5}{8}$ in order that the fraction may be increased three-fold? *Ans.* $2\frac{1}{2}$.

- 27. What number added to both numerator and denominator of the fraction $\frac{3}{4}$ will double the fraction?

28. Find a number of two digits in which the tens' digit is 3 greater than the units' digit, and such that if the number is divided by the sum of the digits, the quotient is 7.

f - 29. In a number of two digits the units' digit exceeds the tens' digit by 4, and when the number is divided by the sum of its digits the quotient is 4. Find the number.

30. Find the two numbers whose sum is 40, such that if the greater is divided by their difference the quotient is 3.

31. The shortest railway route from Boston to Chicago is 166 miles more than 4 times that from Boston to New York; and the shortest route from Boston to Atlanta is 196 miles less than 6 times that from Boston to New York. The distance from Boston to Chicago is 481 miles more than $\frac{1}{2}$ the distance from Boston to Atlanta. Find each of the three distances.

REVIEW QUESTIONS

1. What principle is used in reducing fractions to a common denominator?

2. State from arithmetic the rules for multiplying or dividing a fraction by an integer; by another fraction. State these as formulas by using letters.

3. What is the effect on an equation if both members are multiplied by the same number? How may this be used to clear an equation of fractions?

4. Complete your list of principles stated in symbolic form up to and including Principle XV.

DRILL EXERCISES

1. $\frac{a}{3} + \frac{a+7}{4} - \frac{a-3}{3} = \frac{a+227}{5} - 1.$
2. $(a-2-3c-8+2b)(6-a-c-b+8).$
3. $\frac{a-1}{2} + \frac{a+1}{2} + \frac{a-3}{12} = 2+a.$
4. $a^2b - (3b-8a^2-7) + 3ab^2 - (4ab^2+8-2a^2).$
5. $\frac{a+1}{4} + \frac{a-3}{4} + \frac{a-7}{4} = 2a-26.$
6. $\frac{n+1}{3} + \frac{n+3}{4} + \frac{n-1}{4} = \frac{n+13}{3} + \frac{n-2}{3}.$
7. $16ax+4-(8-8ax-a)-(12ax-13-ax).$
8. Add $15ax^2+3bc^2$, $2bc^2-7ax^2$, and $5+2ax^2-5bc^2$.
9. Add $16-7ab-2a^2+5ab$, $4a^2-2ab$, and $5ab-8$.
10. Add $51x^2y-35+12a^2$, $41-17a^2-57x^2y$, and $3x^2y$.
11. Add $35b^2-13c^2$, $8c^2-3b^2c^2$, and $6b^2-8c^2-9b^2c^2$.
12. Add $19-2x+3x^2b+2b$, $4x^2b+x+5b+8$, and $4x$.
13. $\frac{x}{4} + \frac{x}{8} + \frac{x}{16} - \frac{x}{2} + \frac{x}{32} = -2.$
14. $\frac{y-3}{4} - \frac{y+9}{10} = \frac{y-11}{4} - 2.$
15. $\frac{y-4}{3} + \frac{y+2}{3} + \frac{y+8}{3} = 2y-20.$
16. $\frac{y}{2} + \frac{y+20}{4} + \frac{y+5}{5} = 25.$
17. $\frac{y}{3} - \frac{y+20}{5} + \frac{y-5}{5} + \frac{y-10}{2} = 15.$
18. $\frac{x}{x-1} - \frac{x}{x+1} = \frac{4}{(x-1)(x+1)}.$
19. $-\frac{x-1}{x+2} + \frac{x+1}{x-3} = \frac{27}{(x+2)(x-3)}.$
20. $\frac{2x-1}{x+2} - \frac{4x-1}{2x-3} = \frac{-10}{(x+2)(2x-3)}.$

CHAPTER VI

LITERAL EQUATIONS AND THEIR USES

102. Some of the advantages of algebra over arithmetic in solving problems have been pointed out in the preceding chapters. For instance, the brevity and simplicity of statement secured through the use of letters to represent numbers; the translation of problems into equations; and the clear and logical solution of these equations, step by step.

Another advantage is set forth in the present chapter; namely, the opportunity offered in Algebra to *summarize the solution of a whole class of problems* by solving what is called a **literal equation**, thus obtaining a **formula** which may be used in solving other problems.

For example, in arithmetic we solved many problems obtaining the **interest** when the **principal**, **rate**, and **time** were given. We now see that all of these can be summarized in the one literal equation

$$i = prt.$$

Furthermore, the rules for obtaining the principal, the rate, or the time may now be derived directly from this equation by Principle VI, thus obtaining:

$$p = \frac{i}{rt}, \quad r = \frac{i}{pt}, \quad t = \frac{i}{pr}.$$

Translate each of these formulas into a rule of arithmetic.

103. **Definition.** The process of deriving $p = \frac{i}{rt}$ from $i = prt$ is called **solving** the equation $i = prt$ for p in terms of i , r , and t , or simply **solving the equation for p** .

104. In arithmetic a problem is said to be solved when a *numerical answer* is obtained which satisfies the conditions given. The solutions thus far found in algebra have, for the most part, been of this sort.

It is customary, however, to say that a problem has been solved in the *algebraic sense* when a formula is found which gives complete directions for deriving the numerical answer.

Thus, $p = \frac{i}{rt}$ is a solution for the principal since it states precisely how to find the principal in terms of interest, rate, and time.

105. It is thus seen that from the literal equation $i = prt$ we obtain the complete solution of every problem which calls for any one of these four numbers in terms of the other three.

In modern times machines are extensively used for computation. The algebraic solution of a literal equation *gets the problem ready for the computing machine*, that is, it gets the **formula** which the computer must use.

LITERAL EQUATIONS USED IN SOLVING PROBLEMS

I. PROBLEMS FROM ARITHMETIC

1. If \$700 is invested at 5% simple interest, what is the amount at the end of 5 years? This problem calls for the amount, which is the sum of principal and interest.

If a = amount, then $a = p + i = p + prt = p(1 + rt)$.

Applying this formula, $a = 700(1 + \frac{5}{100} \cdot 5) = 875$.

2. Solve the equation $a = p + prt$ for p in terms a , r , and t . State a rule of arithmetic represented by this solution and make a problem which can be solved by means of it.

3. Solve $a = p + prt$ for r in terms of a , p , and t . State a rule of arithmetic represented by this solution and make a problem which can be solved by means of it.

4. Solve $a = p + prt$ for t in terms of a , p , and r . State a rule of arithmetic represented by this solution and make a problem which can be solved by means of it.

5. A real estate dealer sold a house and lot for \$7500, for which he received a commission of 4%. What was his profit?

Solution. Letting c = commission, b = base, and r = rate, we have

$$c = br.$$

Applying this formula, $c = br = 7500 \cdot \frac{4}{100} = 75 \cdot 4 = 300$.

6. Solve the equation $c = br$ for b in terms of c and r .

7. Solve $c = br$ for r in terms of c and b .

8. State the rules of arithmetic represented by the solutions in Exs. 6 and 7, and make problems to be solved by these rules.

9. How much must I remit to my broker in order that he may buy \$600 worth of bonds and reserve 5% commission?

I must send him both base and commission. Calling this the amount and representing it by a , we have

$$a = b + c = b + br = b(1 + r).$$

Hence,

$$a = 600(1 + \frac{5}{100}) = 630.$$

10. Solve $a = b + br$ for b and translate the result into words.

11. Solve $a = b + br$ for r and translate the result into words.

12. A dealer sold berries for \$18.95, and after deducting a commission of 2% sent the balance to the truck gardener. How much did he remit?

The sum he sent was the difference between the base and the commission; calling this d , we have

$$d = b - c = b - br = b(1 - r).$$

Hence, in this case $d = 18.95(1 - \frac{2}{100}) = 18.57$.

13. Solve the equation $d = b(1 - r)$ for b in terms of d and r . ✕

14. Solve the equation $d = b - br$ for r in terms of b and d . ✕

15. State the rules of arithmetic represented by the solutions in Exs. 13 and 14 and make problems to be solved by these rules.

II. PROBLEMS INVOLVING MOTION

106. In scientific language the distance passed over by a moving body is called the **space**, and the number of units of space traversed is represented by s . The **rate** of uniform motion, that is, the number of units of space traversed in each unit of time, is called the **velocity**, and is represented by v . The number of units of time occupied is represented by t .

Ex. 1. If a train runs 40 miles per hour, how far does it run in 5 hours?

Ex. 2. At a certain temperature sound travels 1080 feet per second. How far does it travel in 5 seconds?

In each of these examples the space passed over is found by multiplying the velocity by the time. Using the symbols s , v , and t , we have

$$s = vt. \quad (1)$$

1. Solve the equation $s = vt$ for t in terms of s and v , and for v in terms of s and t .

Translate each of these formulas into words.

It is to be understood in all problems here considered that the velocity remains the same throughout the period of motion; *e.g.* sound travels just as far in any one second as in any other second of its passage.

2. If sound travels 1080 feet per second, how far does it travel in 6 seconds?

3. If a transcontinental train averages 35 miles per hour, how far does it travel in $2\frac{1}{2}$ days? (Given $v = 35$, $t = 2\frac{1}{2} \cdot 24$, to find s .)

4. A hound runs 23 yards per second and a hare 21 yards per second. If the hound starts 79 yards behind the hare, how long will it require to overtake the hare?

If t is the number of seconds required, then by formula (1) during this time the hound runs $23t$ yards and the hare runs $21t$ yards. Since the hound must run 79 yards farther than the hare, we have: $23t = 21t + 79$.

5. An ocean liner making 21 knots an hour leaves port when a freight boat making 8 knots an hour is already 1240 knots out. In how long a time will the liner overtake the freight?

6. A motor boat starts $7\frac{3}{4}$ miles behind a sailboat and runs 11 miles per hour while the sailboat makes $6\frac{1}{2}$ miles per hour. How long will it require the motor boat to overtake the sailboat?

7. A freight train running 25 miles an hour is 200 miles ahead of an express train running 45 miles an hour. How long before the express will overtake the freight?

8. A bicyclist averaging 12 miles an hour is 52 miles ahead of an automobile running 20 miles an hour. How soon will the automobile overtake him?

9. A and B run a mile race. A runs 18 feet per second and B $17\frac{1}{2}$ feet per second. B has a start of 30 yards. In how many seconds will A overtake B ? Which will win the race?

If in each of the examples 4 to 9 we call the velocity of the first moving object v_1 (read v one) and that of the second v_2 (read v two), then the distance traveled by the first in the required time t is $v_1 t$, and that traveled by the second is $v_2 t$.

Then if n is the distance which the second must go in order to overtake the first, we have

$$v_2 t = v_1 t + n. \quad (2)$$

The solution of (2) for t gives the time required in each problem for the second to overtake the first.

Equation (2) summarizes the solution of all problems like those from 4 to 9.

It is important that formulas (1) and (2) be clearly understood, since they are used very often in problems on motion.

10. A fleet, making 11 knots per hour, is 1240 knots from port when a cruiser, making 19 knots per hour, starts out to overtake it. How long will it require?

Use formula (2).

11. In how many minutes does the minute hand of a clock gain 15 minute spaces on the hour hand?

Using one minute space for the unit of distance and 1 minute as the unit of time, the rates are 1 and $\frac{1}{12}$ respectively, since the hour hand goes $\frac{1}{12}$ of a minute space in 1 minute. Letting t be the number of minutes required, we have, using formula (2),
 $1 \cdot t = \frac{1}{12}t + 15$.

12. In how many minutes after 4 o'clock will the hour and minute hands be together? (Here the minute hand must gain 20 minute spaces.) *Ans.* $21\frac{2}{11}$ min.



13. At what time between 5 and 6 o'clock is the minute hand 15 minute spaces behind the hour hand? At what time is it 15 minute spaces ahead?

Since, at 5 o'clock, it is 25 minute spaces behind the hour hand, in the first case it must gain $25 - 15 = 10$ minute spaces, and in the second case it must gain $25 + 15 = 40$ minute spaces. Make a diagram as in the preceding problem to show both cases.

14. At what time between 9 and 10 o'clock is the minute hand of a clock 30 minute spaces behind the hour hand? At what time are they together?

In each case, starting at 9 o'clock, how much has the minute hand to gain?

15. A fast freight leaves Chicago for New York at 8.30 A.M., averaging 32 miles per hour. At 2.30 P.M. a limited express leaves Chicago over the same road, averaging 55 miles per hour. In how many hours will the express overtake the freight?

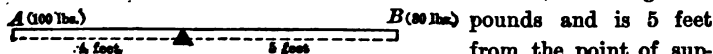
If the express requires t hours to overtake the freight, the latter had been on the way $t + 6$ hours. Then the distance covered by the express is $55t$, and the distance covered by the freight is $32(t + 6)$. As these must be equal, we have $55t = 32(t + 6)$.

16. In a century bicycle race one rider averages $19\frac{1}{2}$ miles per hour, while another, starting 40 minutes later, averages $22\frac{1}{4}$ miles per hour. In how long a time will the latter overtake the former?

III. PROBLEMS INVOLVING THE LEVER

107. Two boys, A and B , play at teeter. They find that the teeter board will balance when equal products are obtained by multiplying the weight of each by his distance from the point of support.

Thus, if B weighs 80 pounds and is 5 feet from the point of support, then A , who weighs 100 pounds, must be 4 feet from this point, since $80 \times 5 = 100 \times 4$.



The teeter board is a certain kind of lever; the point of support is called the **fulcrum**.

In each of the following problems make a diagram similar to the above figure:

1. A and B weigh 90 and 105 pounds respectively. If A is seated 7 feet from the fulcrum, how far is B from that point?
2. Using the same weights as in the preceding problem, if B is $6\frac{1}{2}$ feet from the fulcrum, how far is A from that point?
3. A and B are 5 and 7 feet respectively from the fulcrum. If B weighs 75 pounds, how much does A weigh?
4. A and B weigh 100 and 110 pounds respectively. A places a stone on the board with him so that they balance when B is 6 feet from the fulcrum and A $5\frac{1}{2}$ feet from this point. How heavy is the stone?
5. If the distances from the boys to the fulcrum are respectively d_1 and d_2 , and their weights w_1 and w_2 , then

$$d_1 w_1 = d_2 w_2. \quad (1)$$

This equation is a statement in the language of algebra of a very important law of nature. The law is the result of a very large number of careful experiments. It is a universal custom among scientific men so far as possible to express laws of nature by means of literal equations of this sort.

If any three of the four numbers d_1 , w_1 , d_2 , w_2 , are given, the fourth may be found by means of the equation $d_1 w_1 = d_2 w_2$.

6. Solve $d_1w_1 = d_2w_2$ for each of the four numbers involved in terms of the other three.

7. A and B are seated at the opposite ends of a 13-foot teeter board. Using the weights of problem 1, where must the fulcrum be located so that they shall balance?

If the fulcrum is the distance d from A , then it is $(13 - d)$ from B . Hence, $90d = 105(13 - d)$.

8. A , who weighs 75 pounds, sits 7 feet from the fulcrum, and B , who weighs 105 pounds, sits on the other side. At what distance from the fulcrum should B sit in order to make a balance?

9. A and B together weigh $212\frac{1}{2}$ pounds. They balance when A is 6 feet, and B $6\frac{1}{4}$ feet, from the fulcrum. Find the weight of each.

10. A lever 9 feet long carries weights of 17 and 32 pounds at its ends. Where should the fulcrum be placed so as to make the lever balance?

11. A lever of unknown length is balanced when weights of 30 and 45 pounds are placed on it at opposite ends. Find the length of the lever, if the smaller weight is two feet farther from the fulcrum than the greater.

SUGGESTION. Let x be the distance from the greater weight to the fulcrum.

EXERCISES

Solve the following for each letter in terms of the others:

$$1. F = 32 + \frac{2}{3}C.$$

$$3. s = \frac{n}{2}(a + l).$$

$$2. l = a + (n - 1)d.$$

$$4. s = \frac{a - rl}{1 - r}.$$

Solve each of the following for x :

$$5. ax + 3b = cx + d.$$

$$8. \frac{ax + b}{c} - \frac{bx + c}{d} = 1.$$

$$6. (a - x)(b + x) = x(b - x).$$

$$7. (x + a)(x + b) = (x - c)^2.$$

$$9. \frac{x}{b} + \frac{x}{a - b} = \frac{a}{a + b}.$$

$$10. \frac{x+1}{x-1} = \frac{a+b}{a-b}.$$

$$14. \frac{3ax}{a-b} - 2a = 5x.$$

$$11. \frac{x}{a} + \frac{x}{b} + \frac{x}{c} = 1.$$

$$15. \frac{ax-1}{bx} = \frac{1+bx}{ax} + 1.$$

$$12. \frac{a+bx}{a+b} = \frac{c+dx}{c+d}.$$

$$16. \frac{x-a}{a-b} - \frac{x+a}{a+b} = \frac{2x}{a+b}.$$

$$13. \frac{a+b}{1+x} = \frac{a-b}{1-x}.$$

$$17. \frac{a+x}{a} - \frac{b+x}{b} - \frac{c+x}{c} = 1.$$

$$18. \frac{x+1}{x-1} + \frac{x-1}{x+1} = \frac{2x(x+1)}{(x+1)(x-1)}.$$

$$19. \frac{x+m}{x-m} + \frac{x-m}{x+m} = \frac{2x(x+1)}{(x+m)(x-m)}.$$

$$20. \frac{x+b}{x-a} + \frac{x-c}{x+a} = \frac{2x(x+a)}{(x-a)(x+a)}.$$

$$21. \frac{b}{x-b} + \frac{a}{x+a} = \frac{a}{(x+a)(x-b)}.$$

REVIEW QUESTIONS

1. Make a list of all the rules for interest which are derived from $i = prt$.

2. Make a list of all the rules for percentage which are derived from $a = p + prt$.

3. Make a list of the rules for commission which may be obtained from $c = br$, $a = b + c = b + br$, $d = b - c = b - br$.

4. What problems on motion in this chapter belong to the class whose solutions are summarized by the solutions of the equation,

$$v_2 t = v_1 t + n.$$

5. State fully the meaning of the equation $w_1 d_1 = w_2 d_2$ in connection with the lever.

6. What is the difference in meaning between the solution of a special problem in arithmetic and that of a problem involving a literal equation in algebra?

DRILL EXERCISES

1. Subtract $2a - 6x^2b - 3x + 21$ from $19 - 2x + 3x^2b - 7a$.
2. From $6a - 45 + 8b - 3c + 82cb$ subtract $7b + 18 + 6c$.
3. $(17 + 2a - 3b - 4c)(2 - a + b - c)$.
4. $(13c - 4d + 8e - 3)(c - d)$.
5. $(4xy - 2y - 3x - 2)(y - x + y + 5)$.
6. $(9ax - 3x - 5a - 2x + 4)(5 - x)$.
7. $(9x - 3)(4 - x) + (x - 3)^2 = -8(x + 2)^2 + 94$.
8. $(x + 1)^2 + (x + 2)^2 + (x + 3)^2 = (3x - 1)(x + 12) - 43$.
9. $(2x + 5)(x - 7) - (x - 1)^2 = (x + 1)(x + 2) - 28$.
10. $3(5 - x)^2 - (2x - 1)(x - 1) = (x - 7)(x + 10) + 17x + 50$.
11. $(32 + x)(4x - 1) + (5 - x)^2 + (x - 1)^2 = 6(x + 1)^2 + 194$.
12. $(2x - 7)(5 - x) - (2 - 5x)(1 - x) = -x(7x - 34) - 17$.
13. $\frac{x+8}{2} - \frac{x-9}{12} + \frac{x-17}{6} = \frac{4x-7}{2} + \frac{2x+6}{3} + \frac{5-31x}{12}$.
14. $\frac{3x-1}{6} - \frac{3x+3}{3} + \frac{x-1}{2} = \frac{x+5}{6} + 4x - \frac{20}{3}$.
15. $\frac{x-b}{a} + \frac{x+a}{b} = 2$.
16. $\frac{3x-16}{2} + \frac{21}{x-8} = \frac{6x-11}{4}$.

17. If two numbers differ by d and if the greater of the numbers is x , how do you represent the other?

18. A father is 3 times as old now as his son was 7 years ago. If the son's age now is represented by x , how is the father's age represented?

19. A picture inside the frame is w inches wide and $w + 6$ inches long. The frame is 4 inches wide. Express the area of the frame in terms of w .

20. A picture inside the frame is w inches wide and l inches long. The frame is a inches wide. Express the area of the frame in terms of a , w , and l .

CHAPTER VII

GRAPHIC REPRESENTATION

108. Graphic Representation of Statistics. A graphic representation of the temperatures recorded on a certain day is shown on the next page. The readings were as follows:

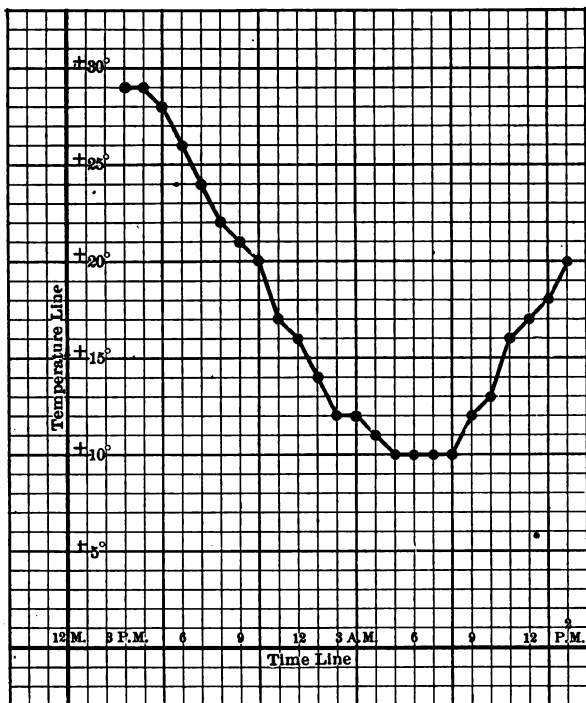
3 P.M.	29°	9 P.M.	21°	3 A.M.	12°	9 A.M.	12°
4 P.M.	29°	10 P.M.	20°	4 A.M.	11°	10 A.M.	13°
5 P.M.	28°	11 P.M.	17°	5 A.M.	10°	11 A.M.	16°
6 P.M.	26°	12 M'T.	16°	6 A.M.	10°	12 Noon	17°
7 P.M.	24°	1 A.M.	14°	7 A.M.	10°	1 P.M.	18°
8 P.M.	22°	2 A.M.	12°	8 A.M.	10°	2 P.M.	20°

In the graph each heavy dot represents the temperature at a certain hour. The distance of a dot to the right of the heavy vertical line indicates the hour of the day counted from noon, and its distance above the heavy horizontal line indicates the thermometer reading at that hour. The lines joining these dots complete the picture representing the gradual changes of temperature from hour to hour.

Graphs of this kind are used in commercial houses to represent variations of sales, fluctuations of prices, etc. They are used by the historian to represent changes in population, fluctuations in mineral productions, etc. In algebra they are used in solving problems and in helping to understand many difficult processes. In the succeeding exercises the cross-ruled paper is essential.

Make a graphic representation of the tables of data on the opposite page:

In each case the number to be represented by one space on the cross-ruled paper should be chosen so as to make the graph go conveniently on a sheet. Thus in Ex. 1 let one small horizontal space represent two years and one vertical space a million of population; and in Ex. 3 let one horizontal space represent one year and one large vertical space one hundred thousand of population.

**EXERCISES**

1. The population of the United States as given by the census reports from 1800 to 1910:

1800 .. 4.3 (million)	1840 .. 17.1	1880 .. 50.2
1810 .. 7.2	1850 .. 23.2	1890 .. 62.6
1820 .. 9.6	1860 .. 31.4	1900 .. 76.3
1830 .. 12.9	1870 .. 38.6	1910 .. 92.0

2. The population of the boroughs now constituting greater New York City:

1800 .. 79 (thousand)	1840 .. 391	1880 .. 1912
1810 .. 119	1850 .. 696	1890 .. 2507
1820 .. 152	1860 .. 1175	1900 .. 3427
1830 .. 242	1870 .. 1478	1910 .. 4767

3. The population of Chicago since 1850:

1850 . . 30 (thousand)	1880 . . 503	1900 . . 1698
1860 . . 109	1890 . . 1100	1910 . . 2185
1870 . . 306		

4. Observe the weather reports in a daily paper and make a graph representing the hourly change of temperature for twenty-four hours.

5. From your own state, city, or town obtain data which you can represent by means of graphs.

GRAPHIC REPRESENTATION OF MOTION

109. A useful picture of the distance traversed by a moving body can be made by a graph similar to the preceding.

E.g. Suppose a man is walking 3 miles per hour. We mark units of time from the starting point to the right along the horizontal reference line, and indicate miles traveled by the number of units measured vertically upward from this line. (See the figure on the opposite page.)

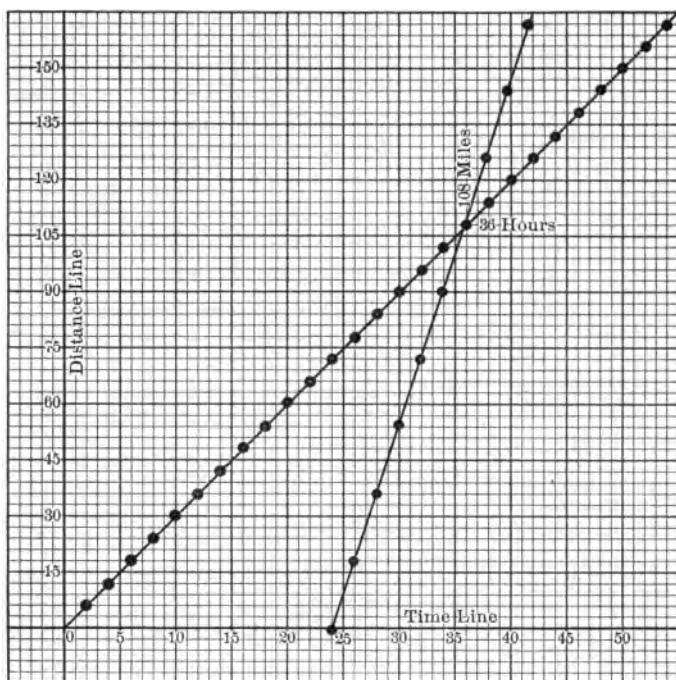
In the figure each horizontal space represents 1 hour, and each vertical space 3 miles. Then in 1 hour he goes 3 miles; in 5 hours, 15 miles; in 10 hours, 30 miles; etc. The dots representing the distances are found to lie on a straight line.

The graph shows at a glance the answers to such questions as: How many miles does he travel in 4 hours? in 13 hours? How long does it take him to go 18 miles? 23 miles?

Again, suppose 24 hours later a second man starts out on a bicycle to overtake the first man, and travels 9 miles an hour. The line drawn from the 24-hour point shows the distance the wheelman travels in any number of hours, counting from his time of starting. The points marked in this line show how far he has gone in 1, 2, 3, 4, 5, 6 hours, etc., namely, 9, 18, 27, 36, 45, 54, etc.

The point where these two lines intersect shows in how many hours after starting the pedestrian is overtaken, and also how far he has gone.

In like manner solve the following by means of graphs, and in each case suggest other questions which may be answered from the graph:



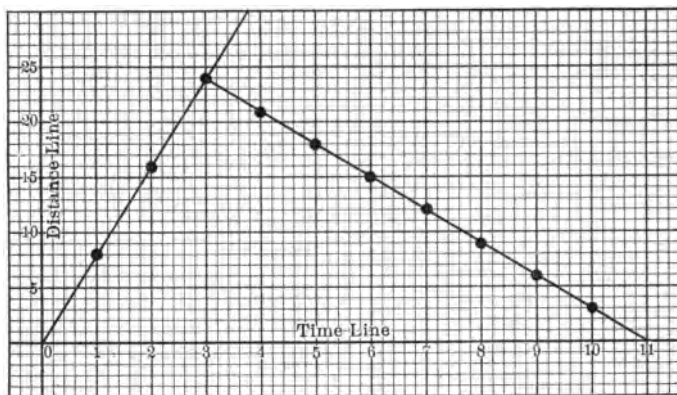
1. *A* starts for a town 12 miles distant, walking 3 miles per hour. $1\frac{1}{2}$ hours later *B* starts for the same place, driving $7\frac{1}{2}$ miles per hour. When does *B* overtake *A*? Where is *A* when *B* reaches town?

2. In a mile race *A* runs 6 yards per second, and *B* 5 yards per second. *B* has a start of 250 yards. Who will win the race? How far in the lead is the winner at the finish?

3. In a century bicycle race *A* averages 17 miles per hour; *B*, who starts 20 minutes later, averages 19 miles per hour. In how many hours will *B* overtake *A*? Who will win the race and where will the loser be when the winner finishes?

Let one vertical small space represent 2 miles, and one horizontal space represent 6 minutes.

110. Illustrative Problem. A man rides a bicycle into the country at the rate of 8 miles per hour. After riding a certain distance the wheel breaks down, and he walks back at the rate of 3 miles per hour. How far does he go before the accident, if he reaches home 11 hours after starting?



In this graph each large horizontal space represents 1 hour, and each small vertical space represents 1 mile. The problem is solved as follows:

(1) Construct the line representing the outward journey at the rate of 8 miles per hour, extending this line indefinitely.

(2) Beginning at the point corresponding to 11 hours, find the points representing his position at each preceding hour. The line connecting these points represents the homeward journey at the rate of 3 miles an hour. Extend this line until it meets the first line. The point where the lines meet represents 3 hours and 24 miles, which is the answer required in the problem.

PROBLEMS

Solve the following problems by means of graphs. In each case prepare a list of questions which may be answered from the graph.

1. A man rows 18 miles per hour down a river, and 2 miles per hour returning. How far down the river can he go if he wishes to return in 10 hours?

2. A man goes from Chicago to Milwaukee on a train running $42\frac{1}{2}$ miles per hour, and returns immediately on a steamer going 17 miles per hour. Find the distance, if the round trip requires 7 hours.

3. A pleasure trip from New York to Atlanta by steamer, and return by rail, occupied 77 hours. Find the distance, if the rate going was 16 miles per hour and returning 40 miles per hour.

Let one small horizontal space represent one hour and one small vertical space 16 miles.

4. *A* invests \$1000 at 5%, and *B* invests \$5000 at 4%. In how many years will the *amount* (principal and interest) of *A*'s investment equal the *interest* on *B*'s investment?

Let one large horizontal space represent one year and one small vertical space \$50. Then the line representing *A*'s *amount* starts at the point marked \$1000, and rises one small vertical space each year. The line representing *B*'s *interest* starts at the zero point and rises four small vertical spaces each year.

5. In how many years will the *interest* on \$6000 equal the *amount* on \$2000 if both are invested at 5%?

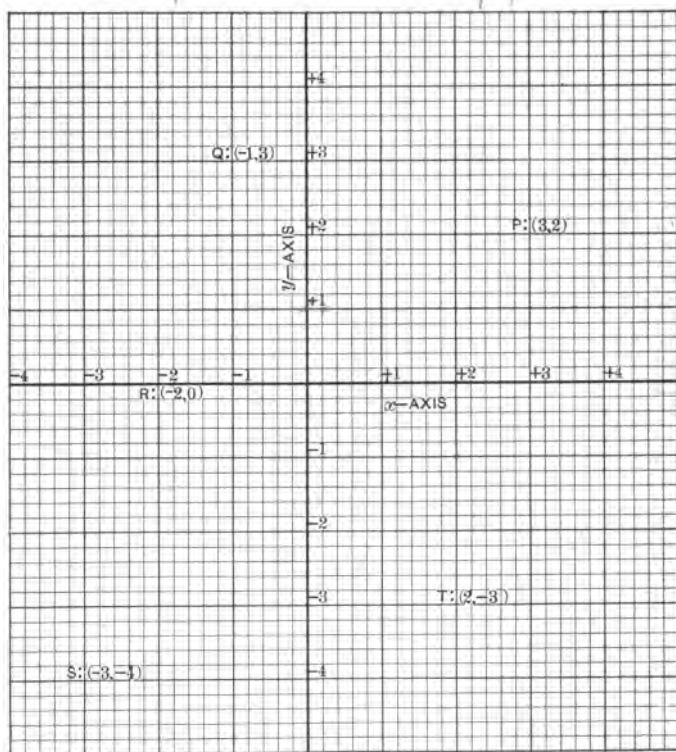
6. *A* invests \$500 at 6% and *B* invests \$1000 at 5%. In how many years will *A*'s interest differ by \$300 from *B*'s?

Let one small vertical space represent \$20. In this case both lines start from the zero point. Find the point on one line which is three large spaces *vertically* above the corresponding point on the other line.

7. A freight train starts from New York to Boston averaging 30 miles per hour. Three hours later a passenger train, averaging 50 miles per hour, starts from New York in the same direction. How long will it require the latter to overtake the freight?

GRAPHIC REPRESENTATION OF EQUATIONS

111. In all the graphs thus far constructed two lines at right angles to each other have been used as reference lines. These lines are called **axes**. The location of a point in the plane of such a pair of axes is completely described by giving its distance and direction from each of the axes. The direction to the right of the vertical axis is denoted by a positive sign, and to the left, by a negative sign; while direction upward from the horizontal axis is positive, and downward, negative.



The horizontal line is usually called the *x*-axis and the vertical line the *y*-axis. The perpendicular distance of any point *P* from the *y*-axis is called the **abscissa** of the point, and its distance from the *x*-axis is called its **ordinate**. The abscissa and ordinate of a point are together called its **coördinates**.

E.g. the abscissa of point *P* in the above figure is 3 and its ordinate 2, or we may say the coördinates of *P* are 3 and 2, and indicate it thus: *P*: (3, 2), writing the abscissa first. In like manner for the other points we write *Q*: (-1, 3), *R*: (-2, 0), *S*: (-3, -4), and *T*: (2, -3).

We see that in this manner every point in the plane corresponds to a pair of numbers, and that every pair of numbers corresponds to a point. This scheme of locating points by two reference lines is already familiar to the pupil in geography, where cities are located by latitude and longitude; that is, by degrees north or south of the equator and east or west of the meridian of Greenwich.

EXERCISES

1. With any convenient scale, locate the following points: (2, 6), (-3, 5), (0, 1), (1, 0), (0, 0), (0, -1), (0, -5), (-5, 0), $(2\frac{1}{2}, 5\frac{1}{2})$, (-4, -8), (3, -10), (-10, 3).

2. Locate the following series of points and then see if a straight line can be drawn through them: (0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (-1, -1), (-2, -2), (-3, -3). Name still other points lying on the same line.

3. Locate the following and connect them by a line: (1, 0), (1, 2), (1, 3), (1, 4), (1, 5), (1, -2), (1, -3), (1, -4), (1, -5). Name other points in this line.

4. Draw the line every one of whose points has its horizontal distance -2, also the line every one of whose points has its vertical distance +3.

5. Locate the following points and see if a straight line can be passed through them: (1, 0), (0, 1), (2, -1), (3, -2), (4, -3), (-1, +2), (-2, 3), (-3, 4), (-4, 5), $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{3}{4}, \frac{1}{4})$. Can you name other points on this line?

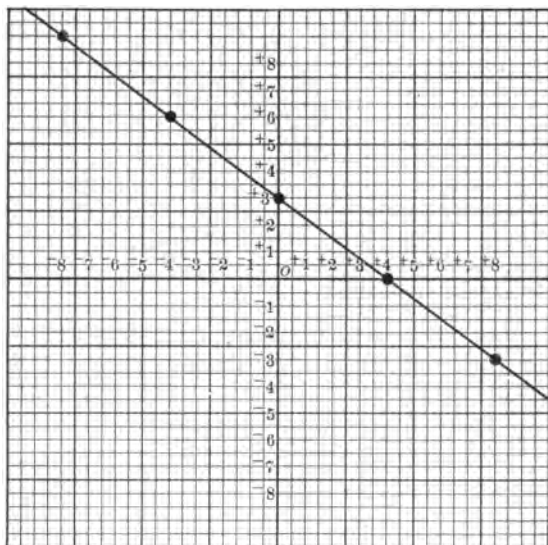
112. In the preceding exercises, in certain cases, a series of points has been found to lie on a **straight line**, as in examples 6, 7, and 8. Evidently this could not happen unless the points were located according to some definite scheme or law.

Illustrative Problem. Locate a series of points whose coördinates are values of x and y which satisfy: $3x + 4y = 12$.

We see that $x = 0, y = 3$, also $x = 4, y = 0$ are pairs of such values. Evidently as many pairs of values as we please may be found by giving any value to x and then solving the equation to find the corresponding value of y . A table may thus be constructed as follows:

Let $x = 0, 4, 8, 12, -4, -8$, etc.

Then find $y = 3, 0, -3, -6, 6, 9$, etc.



These pairs of values for x and y correspond to the points as plotted in the figure, and they are found to lie on a straight line. This line is called the **graph of the equation**.

Let the student find other pairs of numbers which satisfy this equation, and see if the corresponding points lie on this line. Also find the numbers which correspond to any chosen point on this line and see whether they satisfy the equation.

113. We may think of the point whose coördinates are x and y as moving along the line in the graph. Then both x and y will be constantly changing or **varying**, but subject to the relation $3x + 4y = 12$. x and y are therefore called **variables**; and the fixed relation according to which they vary is called a **functional relation**.

One of the graphs on page 105 represents the progress of a man walking 3 miles per hour. As the time varies the point moves along the line, and the distance is seen to vary as the time varies; that is, s varies with t according to the relation $s = 3t$. The variable s is said to be a **function of the variable t** because it is connected with t by a definite relation.

In the equation $s = 3t$ the variable s *increases* as the variable t increases. This may also be seen directly from the graph. In $3x + 4y = 12$, y is seen to *decrease*, as x *increases*.

114. **Definitions.** An equation is said to be of the **first degree** in x and y if it contains each of these letters in such a way that neither x nor y is multiplied by itself or by the other.

E.g. $13x - 5y = 14$ is of the first degree, while $2xy - x = 5$ and $3x - 5y^2 = 13$ are not of the first degree in x and y .

Every equation of the first degree in two variables has for its graph a straight line; hence such an equation is commonly called a **linear equation**.

115. To graph an equation of the first degree it is only necessary to find two points on the graph and draw a straight line through them.

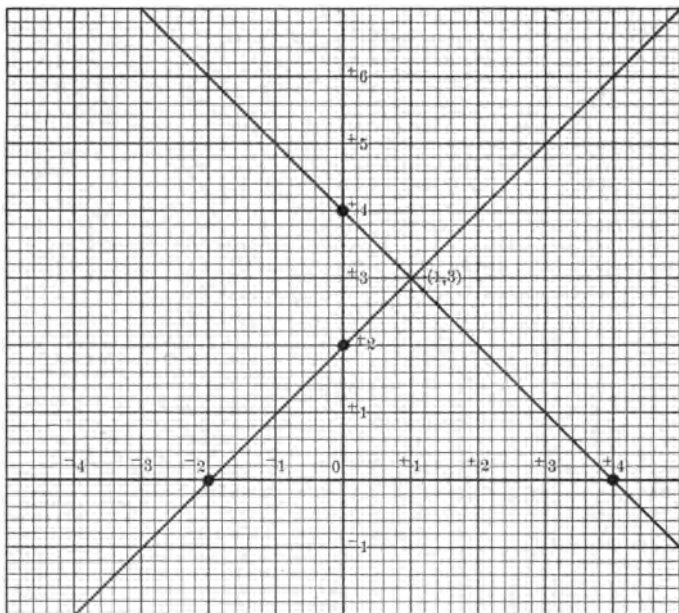
E.g. In graphing the equation $x - y = 5$, we choose $x = 0$ and find $y = -5$, and choose $y = 0$ and find $x = 5$ and plot the points $(0, -5)$ and $(5, 0)$. The line through these points is the one required.

EXERCISES

Construct the graph for each of the following equations. From each graph tell whether y increases or decreases as x increases.

1. $3x + 2y = 1$.
2. $5x - 3y = -3$.
3. $7x + 10y = 2$.
4. $x + 2y = 0$.
5. $5 - 2y = 6x$.
6. $3x + 5y = -15$.
7. $2x - y = 0$.
8. $3x - 4y = 7$.
9. $3x - 4y = -7$.
10. $3x - 4y = -12$.
11. $7y = 9x - 63$.
12. $x = 5y + 3$.

116. **Illustrative Problem.** Graph on the same axes the two equations $x + y = 4$ and $y - x = 2$.



Solution. The two graphs are found to intersect in the point $(1, 3)$. Since the point lies on both lines, its coordinates should satisfy both equations, as indeed they do. Since these lines have only one point in common, there is no other pair of numbers which, when substituted for the variables x and y , can satisfy both equations.

Hence, $x=1$, $y=3$, which is written (1, 3), is called the solution of this pair of equations.

117. **Definition.** These two equations are called **independent** because their graphs are distinct. They are called **simultaneous** because there is at least one pair of values of x and y which satisfy both.

118. Since two straight lines intersect in but one point, it follows that two linear equations which are independent and simultaneous have one and only one solution. The solving of two simultaneous equations may be regarded as *finding the coördinates of the point where their graphs meet*. This may be done by constructing the graphs or by other methods of solution which are considered in the next chapter.

EXERCISES

Graph the following and thus find the solution of each pair of equations:

$$1. \begin{cases} 2x - 3y = 25, \\ x + y = 5. \end{cases}$$

$$6. \begin{cases} y + 3x = 7, \\ 2y + x = -6. \end{cases}$$

$$2. \begin{cases} 5x + 6y = 7, \\ 2x - y = -4. \end{cases}$$

$$7. \begin{cases} x - 2y = 2, \\ 2x - y = -2. \end{cases}$$

$$3. \begin{cases} 5x + 3y = 8, \\ 2x - y = -10. \end{cases}$$

$$8. \begin{cases} 5x - 7y = 21, \\ x - 4y = -1. \end{cases}$$

$$4. \begin{cases} 6x + 8y = 16, \\ 2x - 3y = 11. \end{cases}$$

$$9. \begin{cases} 5x + 2y = 8, \\ 2x - 3y = -12. \end{cases}$$

$$5. \begin{cases} 3x - 4y = 1, \\ 2x - 7y = 5. \end{cases}$$

$$10. \begin{cases} x + 3y = -6, \\ 2x - 4y = -12. \end{cases}$$

119. Not all pairs of equations in two unknowns are independent.

E.g. If we attempt to plot $x + y = 1$ and $2x + 2y = 2$ the graphs will be found to coincide. Such equations are called **dependent**, since one can be derived from the other.

Not all pairs of equations are **simultaneous**.

E.g. If we attempt to plot $x + y = 1$ and $x + y = 2$ the graphs will be found to be parallel and hence they have no point in common. Such equations are called **contradictory**, since it is impossible for $x + y$ to equal both 1 and 2 at the same time.

Historical Note. The representation of equations by means of lines is due to René Descartes (1596–1650). (See also page 50.) This must be regarded as one of the greatest contributions of all time to mathematics. Not only is it possible to represent straight lines by equations as we have here but a very large number of curves of different kinds may be so represented. The points where two curves meet are thus found by solving the equations which represent them. This enables us to use the operations of algebra in solving a large range of problems pertaining to lines and curves.

REVIEW QUESTIONS

1. How may a point in a plane be located by reference to two fixed lines? What are these fixed lines called? What names are given to the distances from the point to the fixed lines? Why are negative numbers needed in order to locate all points in this manner?

2. Draw a pair of axes in a plane and locate the following points: $(5, 0)$, $(-2, 0)$, $(0, 3)$, $(0, -1)$, $(0, 0)$.

3. How many pairs of numbers can be found which satisfy the equation $x - 2y = 6$? State five such pairs and plot the corresponding points. How are these points situated with respect to each other? What can you say of all points corresponding to pairs of numbers which satisfy this equation? What is meant by the graph of an equation?

4. How many pairs of numbers will simultaneously satisfy the two equations $3x + 2y = 7$ and $x + y = 3$? Show by means of a graph that your answer is correct.

5. Is the graph representing the equation $x - y = 3$ limited in either direction? If negative numbers could not be used, would the graph of this equation be limited in either direction?

6. If negative numbers could not be used, how would the graph of $x + y = 3$ be limited?

DRILL EXERCISES

1. From $41a^2 + 7ab - 5c^2 - 9ab$ subtract $8ab + 7c^2 + 50a^2$.
2. From $15 + 2x - 9xy - 3y$ subtract $5xy - 4x - 2y$.
3. Subtract $12 + 8x - 4a - 6c - 18abc$ from $5x - 2a + 3c$.
4. $2 - (71x + 42y - 15x - 64) - (5 - 91x - 2y - 13xy)$.
- 5. $(31 - 2x + 3y - 5)(x + y)$.
6. $8x + (13 + 18x - 6) - (5 - 6x) = 16x + 10$.
7. $5 - (7 - 4x + 2y - 4b) - (8x - 6y - 9 + 2b) + 8a$.
- 8. $8x - (-3 - 2 - 4 - 7) + 5x + (2 + 6 + 4) - (-3x + 2)$.
9. $5y + 2x - (6 - 4x - 5x) - 3y - (4x - 2y) - (-7y + 8)$.
10. $35y - (41x - 16 - 12y) + 5x + (-6 + 46y - 18x)$.
11. $(x - 1)^2 - (x - 8)(2x - 1) = -x^2 + 98$.
12. $(7 + x)(x - 4) + (1 - x)^2 = -23 + 2x^2$.
13. $(12 - 4x)(2 - x) - 4(1 + x)^2 = 5x + 119$.
14. $(x - 17)(59 - 2x) - (1 - x)^2 = (6 - 3x)(x - 2) + 384$.
- 15. $(3x - 2) + (x - 1)^2 + (x - 2)^2 = 2(x - 1)(x - 2) + 5$.
- 16. $(6 - 3x)(2 + x) + 16(x - 1)^2 = 13(x + 4)^2 + 364$.
- 17. $\frac{1}{x - b} + \frac{1}{x - a} = \frac{2}{x - c}$.
- 18. $\frac{7}{x - a} - \frac{3}{x + a} = \frac{4}{x + c}$.
19. Solve for each letter in terms of the others:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$$

20. The sum of two numbers is s and one of them is x . How do you represent the other?

21. The difference between two numbers is d and one of them is x . How do you represent the other?

22. How do you represent the area of a rectangle whose width and length are w and l feet respectively? How do you represent the area of a rectangle which is a feet wider and b feet longer than this one?

CHAPTER VIII

SIMULTANEOUS LINEAR EQUATIONS

120. Simultaneous linear equations have already been considered in connection with graphs. In this chapter such equations will be solved by the algebraic method called **elimination**. The process consists in combining the equations in such a way as to *get rid of one* of the unknown numbers.

ELIMINATION BY ADDITION OR SUBTRACTION

121. Illustrative Examples. Solve

$$\begin{cases} x + 2y = 7, & (1) \\ 3x - 2y = 5. & (2) \end{cases}$$

Adding the members of these equations, $+2y$ and $-2y$ cancel. Hence, $4x = 12$, and $x = 3$.

Substituting in (1), $3 + 2y = 7$, and $y = 2$.

Verify this by drawing the graphs, and also by substituting $x = 3$, $y = 2$, in (1) and (2).

If one of the variables does not already have the same coefficient in both equations, we proceed as follows:

Solve the equations

$$\begin{cases} 7x + 3y = 4 - y + 4x, & (1) \\ 3x - y = 4y - 2 - x. & (2) \end{cases}$$

$$\text{From (1) by } A, S, \quad 3x + 4y = 4. \quad (3)$$

$$\text{From (2) by } A, S, \quad 4x - 5y = -2. \quad (4)$$

$$(3) \cdot 4 \quad 12x + 16y = 16. \quad (5)$$

$$(4) \cdot 3 \quad 12x - 15y = -6. \quad (6)$$

$$(5) - (6) \quad 31y = 22. \quad (7)$$

$$D \mid 31 \quad y = \frac{22}{31}. \quad (8)$$

$$\text{Substituting in (3),} \quad x = \frac{11}{31}. \quad (9)$$

Hence, the solution is $\frac{11}{31}, \frac{22}{31}$.

(3) · 4 means that the members of equation (3) were multiplied by 4. (5) - (6) means that the members of equation (6) were subtracted from those of (5).

The process used in the solution just given is called **elimination by addition or subtraction**.

Make a rule for solving a pair of equations by this process.

EXERCISES

Solve the following pairs of equations by addition or subtraction. Substitute the results in the given equations in each case to test the accuracy of the solution.

- | | |
|--------------------------------------------------------------------|----------------------------------------------------------------------------------|
| 1. $\begin{cases} 2x + 3y = 22, \\ x - y = 1. \end{cases}$ | 11. $\begin{cases} 7m = 2n - 3, \\ 19n = 6m + 89. \end{cases}$ |
| 2. $\begin{cases} 5x - 2y = 21, \\ x - y = 6. \end{cases}$ | 12. $\begin{cases} 15k = 10 - 20l, \\ 25k - 30l = 80. \end{cases}$ |
| 3. $\begin{cases} 6x + 30 = 8y, \\ 3y + 17 = 2 - 3x. \end{cases}$ | 13. $\begin{cases} 6c + 15d = -6, \\ 21d - 8c = -74. \end{cases}$ |
| 4. $\begin{cases} 8x - 4y = 12x, \\ 4x + 2y = 3 + 4y. \end{cases}$ | 14. $\begin{cases} 2x - 3y = 4, \\ 2y - 3x = -21. \end{cases}$ |
| 5. $\begin{cases} x + 6y = 2x - 16, \\ 3x - 2y = 24. \end{cases}$ | 15. $\begin{cases} u + v = 27, \\ \frac{2}{3}v = 19 - \frac{3}{4}u. \end{cases}$ |
| 6. $\begin{cases} 5x + 10y = -7, \\ 2x + 5y = -2. \end{cases}$ | 16. $\begin{cases} 7a = 1 + 10y, \\ 16y = 10a - 1. \end{cases}$ |
| 7. $\begin{cases} 5x + 3y = -2, \\ 3x + 2y = -1. \end{cases}$ | 17. $\begin{cases} 28x + 14y = 23, \\ 14x - 14y = 1. \end{cases}$ |
| 8. $\begin{cases} 3a + 7b = 7, \\ 5a + 3b = 29. \end{cases}$ | 18. $\begin{cases} 5x + 2y = x + 18, \\ 2x + 3y = 3x + 27. \end{cases}$ |
| 9. $\begin{cases} r = 3s - 19, \\ s = 3r - 23. \end{cases}$ | 19. $\begin{cases} 7y - x = x - 17, \\ 2y + 3x = 38. \end{cases}$ |
| 10. $\begin{cases} 2p = 5q - 16, \\ 7q = -3p + 5. \end{cases}$ | 20. $\begin{cases} 6x + 2y = -2, \\ x - 4y = -35. \end{cases}$ |

ELIMINATION BY SUBSTITUTION

122. Illustrative Problem. Solve the equations:

$$\begin{cases} 2x + 3y = 13. & (1) \end{cases}$$

$$\begin{cases} 5x - 6y = -8. & (2) \end{cases}$$

From (1) by *S* and *D*, $y = \frac{13 - 2x}{3}. \quad (3)$

Substituting in (2), $5x - \frac{6(13 - 2x)}{3} = -8. \quad (4)$

By *F*, *V*, $5x - 2(13 - 2x) = -8. \quad (5)$

By *IV*, *VII*, $5x - 26 + 4x = -8. \quad (6)$

By *I*, *A*, $9x = 18. \quad (7)$

By *D*, $x = 2. \quad (8)$

From (3), $y = \frac{13 - 2 \cdot 2}{3} = 3. \quad (9)$

Verify this by substituting these values of *x* and *y* in (1) and (2).

The process here used is called **elimination by substitution**.

Make a rule for solving a pair of equations by this process.

This process is usually convenient when one of the variables has a coefficient unity in one equation and not in the others, since in that case no fractions are introduced by the substitution.

EXERCISES

$$1. \begin{cases} x + 2y = 4, \\ 2x + y = 5. \end{cases} \quad 6. \begin{cases} x - y = 37, \\ 2x + 3y = 31x + 13y. \end{cases}$$

$$2. \begin{cases} 3x - y = 5, \\ 5x + 2y = 23. \end{cases} \quad 7. \begin{cases} 2x - y = y + 6, \\ x + 2y = 4y + 3. \end{cases}$$

$$3. \begin{cases} 2x + y = 3, \\ 3x - 7y = 30. \end{cases} \quad 8. \begin{cases} 5x + 3y \div 0, \\ 2x + y = 1. \end{cases}$$

$$4. \begin{cases} 5y + x = 7, \\ 5x - 3y = 4 - 2x + 7. \end{cases} \quad 9. \begin{cases} 3x - 2y = 3, \\ 2x + 3y = 6x - 1 \end{cases}$$

$$5. \begin{cases} 5x + 8y = -1, \\ 6y - x = 4y - 7. \end{cases} \quad 10. \begin{cases} 5x - 3y = 0, \\ 2x - 6y = -x. \end{cases}$$

EXERCISES

Solve by either method of elimination :

✓ 1. $\begin{cases} x+y=4, \\ x-y=10. \end{cases}$

✓ 2. $\begin{cases} x-y=-3, \\ x+4y=12. \end{cases}$

✓ 3. $\begin{cases} 2x+3y=5, \\ 7x-5y=33. \end{cases}$

✓ 4. $\begin{cases} 3x-4y=8, \\ 4x+3y=-6. \end{cases}$

✓ 5. $\begin{cases} 2x-4y=8, \\ 3x+2y=4. \end{cases}$

- 6. $\begin{cases} 3x-4y=8, \\ 2x+3y=11. \end{cases}$

- 7. $\begin{cases} 4y-2x=3, \\ 2y+5x=6. \end{cases}$

- 8. $\begin{cases} 3x-y=2x-1, \\ 12x+y=14. \end{cases}$

- 9. $\begin{cases} 2x+3y=5, \\ 6x+14y=0. \end{cases}$

- 10. $\begin{cases} 4x+3y=5, \\ 7x-2y=74. \end{cases}$

11. $\begin{cases} 6y+2x=11, \\ 3y+12x=18. \end{cases}$

12. $\begin{cases} 4y+9x=-5, \\ x+y=-5. \end{cases}$

13. $\begin{cases} 3y+5x=12+2x, \\ 17x-y=4y-20. \end{cases}$

14. $\begin{cases} 6+x+y=2x-1, \\ 3y+x=6y+9. \end{cases}$

15. $\begin{cases} y+5x=2x+5, \\ 2y-3x=19. \end{cases}$

16. $\begin{cases} 6x+2y=23, \\ 10x-5y=21. \end{cases}$

Ans. $x=3\frac{7}{8}, y=2\frac{3}{8}.$

17. $\begin{cases} 3x-7y=15, \\ 5x+4y=11. \end{cases}$

Ans. $x=1\frac{3}{4}, y=-\frac{1}{4}.$

18. $\begin{cases} 7x-4y=3, \\ 5x+8y=6. \end{cases}$

19. $\begin{cases} 12y-10x=-6, \\ 7y+x=99. \end{cases}$

20. $\begin{cases} 7x-3y=-7, \\ 5y-9x=1. \end{cases}$

21. $\begin{cases} 7x+4y=3, \\ 2x+3y=25. \end{cases}$

22. $\begin{cases} 34x+70y=4, \\ 5x-8y=-36. \end{cases}$

23. $\begin{cases} 7x+9y=8, \\ 2x-3y=21, \end{cases}$

24. $\begin{cases} 8x+4y=49, \\ 5x-8y=28, \end{cases}$

123. The equations thus far given have for the most part been written in a standard form, $ax + by = c$, in which all the terms containing x are collected, likewise those containing y , and those which contain neither variable. When the equations are not given in this form, they should be so reduced at the outset, as in the following.

Example: Solve $\begin{cases} \frac{7y-4}{5} + \frac{2x-3}{2} = \frac{3}{2} \\ \frac{5x-2}{3} + \frac{2y+1}{5} = 2. \end{cases}$ (1)

Solution. (1) $\cdot 10$, $14y - 8 + 10x - 15 = 15$, (3)

Transposing in (3), $14y + 10x = 38$. (4)

(4) $\div 2$, $7y + 5x = 19$. (5)

(2) $\cdot 15$, $25x - 10 + 6y + 3 = 30$. (6)

Transposing in (6), $25x + 6y = 37$. (7)

(5) $\cdot 5$, $25x + 35y = 95$. (8)

(8) $-$ (7), $29y = 58$. (9)

(9) $\div 29$, $y = 2$. (10)

Substituting in (7), $x = 1$.

Check. Substitute $x = 1$, $y = 2$ in (1) and (2) and see that each is satisfied.

EXERCISES

After reducing each of the following pairs of equations to the standard form, solve by means of either process of elimination:

1. $\begin{cases} x - 14 = 7y, \\ 6y + 1 = x. \end{cases}$

2. $\begin{cases} 16x - 3y = 7x, \\ 4y = 7x + 5. \end{cases}$

3. $\begin{cases} r + 1 = -4s, \\ 2s = 13 - 5r. \end{cases}$

4. $\begin{cases} m = \frac{1-8n}{5}, \\ 3m + 5n = 1. \end{cases}$

5. $\begin{cases} m - n = 16, \\ 3m = 8 - 2n. \end{cases}$

$$6. \begin{cases} \frac{7x-15}{3} = y, \\ 2x-y=3. \end{cases}$$

$$7. \begin{cases} \frac{x-3}{5y} = -2, \\ x+7y=6. \end{cases}$$

$$8. \begin{cases} 3x-5=-y, \\ 8y+76=5x. \end{cases}$$

$$9. \begin{cases} a+4b=14, \\ 3a-b=14. \end{cases}$$

$$\text{Ans. } a=5\frac{5}{8}, b=2\frac{2}{8}.$$

$$10. \begin{cases} \frac{x+y}{2} + \frac{x-y}{2} = 10, \\ 2x-y=16. \end{cases}$$

$$11. \begin{cases} \frac{x-y}{5} + \frac{x+y}{3} = 8, \\ \frac{2x-y}{2} - \frac{3y-x}{4} = 12\frac{1}{2}. \end{cases}$$

$$12. \begin{cases} \frac{7m+8}{5} - \frac{7n-1}{4} = -2, \\ \frac{2m-4}{4} + \frac{n-1}{3} = -\frac{1}{3}. \end{cases}$$

$$13. \begin{cases} \frac{x-3}{4} + \frac{y+8}{5} = 2, \\ \frac{x+7}{2} + \frac{2y-4}{7} = 5. \end{cases}$$

$$14. \begin{cases} \frac{8a-3}{9} + \frac{5b-2}{3} = 13, \\ \frac{2a+7}{5} - \frac{3b+10}{10} = -3\frac{1}{2}. \end{cases}$$

$$15. \begin{cases} \frac{7y-4}{5} + \frac{2x-3}{2} = 7, \\ \frac{6x-3}{5} + \frac{2y+1}{5} = 7. \end{cases}$$

$$\text{Ans. } x=5\frac{3}{8}, y=2\frac{5}{8}.$$

$$16. \begin{cases} \frac{3y+7}{2} - \frac{5x-7}{3} = 10, \\ \frac{2x-4}{3} - \frac{2y-1}{4} = -2\frac{1}{4}. \end{cases}$$

$$17. \begin{cases} \frac{5+3p}{7} - \frac{5q-2}{4} = -2, \\ 6p+8q=108. \end{cases}$$

$$18. \begin{cases} 3x-2y=4, \\ \frac{2x-1}{5} - \frac{7y-4}{3} = -19 \end{cases}$$

$$19. \begin{cases} 5x+7y=89\frac{1}{2}, \\ \frac{2x-4}{3} + \frac{6y-1}{5} = 13\frac{1}{2}. \end{cases}$$

$$20. \begin{cases} 32x-9y=299, \\ \frac{2x-5}{7} - \frac{3y-1}{2} = -16. \end{cases}$$

SIMULTANEOUS LITERAL EQUATIONS

Examples. 1. The sum of two numbers is 35 and their difference is 5. What are the numbers?

Let x represent one number and y the other. Form two equations and solve them.

2. The sum of two numbers is 48 and their difference is 24. What are the numbers?

3. The sum of two numbers is $41\frac{1}{2}$ and their difference is $23\frac{1}{2}$. What are the numbers?

4. The sum of two numbers is 8590 and their difference is 3480. What are the numbers?

5. If the sum of two numbers is s and their difference is d , find the number.

Solution. Let x represent one of the numbers and y the other.

$$\text{Then,} \quad x + y = s, \quad (1)$$

$$x - y = d. \quad (2)$$

$$\text{Solving, we get} \quad x = \frac{s+d}{2} = \frac{s}{2} + \frac{d}{2},$$

$$\text{and} \quad y = \frac{s-d}{2} = \frac{s}{2} - \frac{d}{2}.$$

Translated into words these results are:

Given the sum of any two numbers and their difference. Then one of the numbers is half their sum plus half their difference and the other is half their sum minus half their difference.

6. Test this general solution by applying to the following:

$$s = 48, \quad d = 24.$$

$$s = 40, \quad d = 52.$$

$$s = 8590, \quad d = 348.$$

$$s = 38, \quad d = 50.$$

In the following x and y are the unknowns. Solve for them in terms of the other letters.

$$\times 7. \begin{cases} ax + by = 1, \\ cx - dy = 1. \end{cases} \quad \begin{matrix} x = \frac{a+d}{c+d} \\ y = \frac{a-d}{c-d} \end{matrix} \quad 9. \begin{cases} mx + 2ny = k, \\ 3 + 2mx = +ny. \end{cases}$$

$$8. \begin{cases} 2x - 3y = c - d, \\ 3x - 2y = c + d. \end{cases} \quad \begin{matrix} x = \frac{a+d}{c+d} \\ y = \frac{a-d}{c-d} \end{matrix} \quad 10. \begin{cases} x + y = a, \\ \frac{x}{a+b} - \frac{y}{a-b} = 1. \end{cases}$$

DRILL EXERCISES

1. $(x-2)^2 - (x-1)(x+2) = 6 - 5x$.
2. $(x-2)(x+2) + (3x-1)(2-x) = (x-2)(5-2x)$.
3. $(x-3)^2 + (2x+5)^2 = (5x-3)(x+5) - 7$.
4. $(2-x)^2 - (2x-1)^2 = (-3x+1)(4+x) - 4$.
5. $\begin{cases} 2x - 3y - 1 = 0, \\ 5x + 2y = 12. \end{cases}$

In the following x and y are the unknowns. Solve for them in terms of the other letters.

6. $\begin{cases} x + y = a, \\ x - y = b. \end{cases}$
8. $\begin{cases} \frac{x}{a} + \frac{y}{b} = c, \\ \frac{x}{a} - \frac{y}{b} = d. \end{cases}$
7. $\begin{cases} ax + by = c, \\ ax - by = d. \end{cases}$
9. $(x-a)^2 + (x-b)^2 = 2(x-c)^2$.
10. $ax - a(b-x) + ac = 3ab$.
11. $\frac{2x-a}{x+b} - \frac{2x-b}{2x+a} = 1$.
12. $\frac{3}{x+a} + \frac{4}{x-b} = \frac{7}{x-a}$.
13. $2ax - 4by + 2cx - (cx - 3ax - 4by)$.
14. Find the value of $\frac{x^2 - 3cx + 8c}{c^2 + ab - 15}$ for $x=1$, $a=2$, $b=3$, $c=4$.

15. A father is now twice the age of his son. If x represents the son's age now, express twice the sum of their ages 5 years ago.

16. One number is 3 less than 4 times another number. If x represents one of the numbers express one third the sum of the numbers.

17. If h is the digit in hundreds' place, t the digit in tens' place, and u the digit in units' place of a certain number, express the number obtained by inverting the order of the digits of the given number.

18. h , t , and u are the digits in hundreds', tens', and units' places of a number. Express the number obtained by increasing each digit by 2.

124. Many problems may be solved, using either one or two unknowns.

Ex. Find two numbers, whose sum is 20, such that when one of them is subtracted from twice the other, the remainder is 16.

(a) *Using one unknown.* Let x represent one number. Then $20 - x$ is the other number, and the equation is $2x - (20 - x) = 16$.

(b) *Using two unknowns.* Let x and y represent the two numbers. Then,

$$\begin{cases} x + y = 20, \\ 2x - y = 16. \end{cases}$$

The translation of problems into equations is usually easier when more than one unknown is permitted. This is due to the fact that in this case each of the given relations between the numbers is put down as a *separate equation*.

PROBLEMS INVOLVING TWO VARIABLES

1. If w and l are the width and length of a rectangle, express its area and also its perimeter in terms of w and l .

2. If the width of the rectangle in the preceding is increased by 10 and its length by 20, express its new perimeter and also its new area in terms of w and l .

3. If x and y represent the ages of a father and son respectively, represent the sum of their ages 5 years ago in terms of x and y ; also the difference of their ages 8 years hence.

4. If a number consisting of two digits is increased by 15 by changing the order of its digits, which is greater, the digit in tens' or in units' place?

5. If a number is decreased by changing the order of its digits, which is greater, the digit in tens' or in units' place?

In solving the following problems, use two variables in each case:

6. A rectangular field is 32 rods longer than it is wide. The length of the fence around it is 308 rods. Find the dimensions of the field.

7. Find two numbers such that 7 times the first plus 4 times the second equals 37; while 3 times the first plus 9 times the second equals 45.

8. A certain sum of money was invested at 5% interest and another sum at 6%, the two investments yielding \$980 per annum. If the first sum had been invested at 6% and the second at 5%, the annual income would be \$1000. Find each sum invested.

9. The combined distance from the sun to Jupiter and from the sun to Saturn is 1369 million miles. Saturn is 403 million miles farther from the sun than Jupiter. Find the distance from the sun to each planet.

10. Find two numbers such that 7 times the first plus 9 times the second equals 116, and 8 times the first minus 4 times the second equals 4.

11. The sum of two numbers is 108. 8 times one of the numbers is 9 greater than the other number. Find the numbers.

12. Two investments of \$24,000 and \$16,000 respectively yield a combined income of \$840. The rate of interest on the larger investment is 1% greater than that on the other. Find the two rates of interest.

13. A father is twice as old as his son. Twenty years ago the father was six times as old as his son. How old is each now?

14. If the length of a rectangle is increased by 3 feet and its width decreased by 1 foot, its area is increased by 3 square feet. If the length is increased by 4 feet and the width decreased by 2 feet, the area is decreased by 3 square feet. What are the dimensions of the rectangle?

Note that if w and l are the original width and length of the rectangle, the term lw will cancel out of both equations.

15. A steamer on the Mississippi makes 6 miles per hour going against the current and $19\frac{1}{2}$ miles per hour going with the current. What is the rate of the current and at what rate can the steamer go in still water?

16. *A* starts at 7 A.M. for a walk in the country. At 10 A.M. *B* starts on horseback to overtake *A*, which he does at 1 P.M. If the rate of *B* had been two miles per hour less, he would have overtaken *A* at 4 P.M. At what rate does each travel?

17. A camping party sends a messenger with mail to the nearest post office at 5 A.M. At 8 A.M. another messenger is sent out to overtake the first, which he does in $2\frac{1}{4}$ hours. If the second messenger travels 5 miles per hour faster than the first, what is the rate of each?

18. There are two numbers such that 3 times the greater is 18 times their difference, and 4 times the smaller is 4 less than twice the sum of the two. What are the numbers?

19. A picture is 3 inches longer than it is wide. The frame 4 inches wide has an area of 360 square inches. What are the dimensions of the picture?

20. The difference between 2 sides of a rectangular wheat field is 30 rods. A farmer cuts a strip 5 rods wide around the field, and finds the area of this strip to be $7\frac{1}{2}$ acres. What are the dimensions of the field?

21. The sum of the length and width of a certain field is 260 rods. If 20 rods are added to the length and 10 rods to the width, the area will be increased by 3800 square rods. What are the dimensions of the field?

22. In a number consisting of two digits the sum of the digits is 12. If the order of the digits is reversed, the number is increased by 36. What is the number?

23. A bird attempting to fly against the wind is blown backward at the rate of $7\frac{1}{2}$ miles per hour. Flying with a wind $\frac{1}{4}$ as strong, the bird makes 48 miles an hour. Find the rate of the wind and the rate at which the bird can fly in calm weather.

24. There is a number whose two digits differ by 2. If the digit in units' place is multiplied by 3 and the digit in tens' place is multiplied by 2, the number is increased by 44. Find the number, the tens' digit being the larger.

25. In a number consisting of two digits one digit is equal to twice their difference. If the order of the digits is reversed, the number is increased by 18. Find the number.

26. If the length of a rectangle is doubled and 8 inches added to the width, the area of the resulting rectangle is 180 square inches greater than twice the original area. If the length and width of the rectangle differ by 10, what are its dimensions?

27. There is a number consisting of three digits, those in tens' and units' places being the same. The digit in hundreds' place is 4 times that in units' place. If the order of the digits is reversed, the number is decreased by 594. What is the number?

28. A man rowing against a tidal current drifts back $2\frac{1}{2}$ miles per hour. Rowing with this current, he can make $14\frac{1}{2}$ miles per hour. How fast does he row in still water and how swift is the current?

29. Flying against a wind a bird makes 28 miles per hour, and flying with a wind whose velocity is $2\frac{2}{3}$ times as great, the bird makes 46 miles per hour. What is the velocity of the wind and at what rate does the bird fly in calm weather?

30. A freight train leaves Chicago for St. Paul at 11 A.M. At 3 and 5 P.M. respectively of the same day two passenger trains leave Chicago over the same road. The first overtakes the freight at 7 P.M. the same day, and the other, which runs 10 miles per hour slower, at 3 A.M. the next day. What is the speed of each?

31. Two boys, *A* and *B*, trying to determine their respective weights, find that they balance on a teeter board when *B* is 6 feet and *A* 5 feet from the fulcrum. If *B* places a 30-pound weight on the board beside him, they balance when *B* is 4 and *A* 5 feet from the fulcrum. How heavy is each boy?

32. \$10,000 and \$8000 are invested at different rates of interest, yielding together an annual income of \$820. If the first investment were \$12,000 and the second \$6000, the yearly income would be \$840. Find the rates of interest.

SIMULTANEOUS EQUATIONS IN THREE VARIABLES

125. Illustrative Problem. Three men were discussing their ages and found that the sum of their ages was 90 years. If the age of the first were doubled and that of the second trebled, the aggregate of the three ages would then be 170. If the ages of the second and third were each doubled, the sum of the three would be 160. Find the age of each.

Solution. Let x , y , and z represent the number of years in their ages in the order named.

$$\text{Then,} \quad x + y + z = 90, \quad (1)$$

$$2x + 3y + z = 170, \quad (2)$$

$$\text{and} \quad x + 2y + 2z = 160. \quad (3)$$

If we subtract (1) from (2), we obtain a new equation from which z is eliminated.

$$\text{I.e.} \quad x + 2y = 80. \quad (4)$$

Again, multiplying (2) by 2 and subtracting (3),

$$3x + 4y = 180. \quad (5)$$

(4) and (5) are two equations in the two variables x and y . Solving these by eliminating y , we find $x = 20$. (6)

$$\text{Substituting } x = 20 \text{ in (4),} \quad y = 30. \quad (7)$$

$$\text{Substituting } x \text{ and } y \text{ in (1),} \quad z = 40. \quad (8)$$

Check by showing that the values of x , y , and z satisfy the original equations and the conditions of the problem.

The values of x , y , and z as thus found constitute the **solution of the given system** of equations.

Evidently x could have been eliminated first, using (1), (2), and (1), (3), giving a new set of two equations in y and z . Let the student find the solution in this manner.

Also find the solution by first eliminating y , using (1), (2), and (2), (3), getting two equations in x and z , from which the values of x and z can be found.

126. Definition. An equation is said to be of the **first degree** in x , y , and z if it contains these letters in such a way that no one of them is multiplied by itself or by one of the others (§ 115).

The fact that the solutions are found to be the same, no matter in what order the equations are combined, indicates that *a system of three independent and simultaneous equations of the first degree in three variables has one and only one solution.*

As in the case of two equations, each should be first reduced to a standard form in which all the terms containing a given variable are collected and united and all fractions removed. Make a rule for solving a system of three equations each of the first degree in three variables.

EXERCISES

Solve the following systems, and check the results:

- | | |
|-------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------|
| 1. $\begin{cases} 2x - y + z = 18, \\ x - 2y + 3z = 10, \\ 3x + y - 4z = 20. \end{cases}$ | 7. $\begin{cases} x + y + z = 1, \\ x + 3y + 2z = 8, \\ 2x + 8y - 3z = 15. \end{cases}$ |
| 2. $\begin{cases} 5x - 3y + z = 15, \\ x + 3y - z = 3, \\ 2x - y + z = 8. \end{cases}$ | 8. $\begin{cases} 2x - 3y + z = 5, \\ 3x + 2y - z = 5, \\ x + y + z = 3. \end{cases}$ |
| 3. $\begin{cases} 4x + 2y + z = 13, \\ x - y + z = 4, \\ x + 2y - z = 1. \end{cases}$ | 9. $\begin{cases} x + y + z = 6, \\ 3x - 2y - z = 13, \\ 2x - y + 3z = 26. \end{cases}$ |
| 4. $\begin{cases} 6x + 4y - 4z = -4, \\ 4x - 2y + 8z = 0, \\ x + y + z = 4. \end{cases}$ | 10. $\begin{cases} x + y + z = 6, \\ 4x - y - z = -1, \\ 2x + y - 3z = -6. \end{cases}$ |
| 5. $\begin{cases} x + 2y + 3z = 5, \\ 4x - 3y - z = 5, \\ x + y + z = 2. \end{cases}$ | 11. $\begin{cases} 2x - 3y - 4z = 17, \\ 4x - 4y + 2z = -10, \\ 7x + 7y + 5z = 17. \end{cases}$ |
| 6. $\begin{cases} 2x - 8y + 3z = 2, \\ x - 4y + 5z = 1, \\ 3x - 10y - z = 5. \end{cases}$ | 12. $\begin{cases} x + y + z = 0, \\ 5x + 3y + 4z = -1, \\ 2x - 7y + 6z = 21. \end{cases}$ |

PROBLEMS INVOLVING THREE VARIABLES

127. Illustrative Problem. A broker invested a total of \$15,000 in the street railway bonds of three cities, the first investment yielding 3%, the second $3\frac{1}{2}\%$, and the third 4%, thus securing an income of \$535 per year. If the second investment was one-half the sum of the other two, what was the amount of each?

Solution. Suppose x dollars were invested at 3%, y dollars at $3\frac{1}{2}\%$, and z dollars at 4%.

$$\text{Then,} \quad x + y + z = 15000, \quad (1)$$

$$.03x + .035y + .04z = 535, \quad (2)$$

$$\text{and} \quad x + z = 2y. \quad (3)$$

$$\text{From (3),} \quad x - 2y + z = 0. \quad (4)$$

$$\text{Subtract (4) from (1),} \quad 3y = 15000, \quad (5)$$

$$\text{and} \quad y = 5000. \quad (6)$$

$$\text{From (1), by } M, \quad .035x + .035y + .035z = 525. \quad (7)$$

$$\text{Subtract (7) from (2),} \quad -.005x + .005z = 10. \quad (8)$$

$$\text{Divide (8) by .005,} \quad -x + z = 2000. \quad (9)$$

$$\text{Substitute (6) in (4),} \quad x + z = 10000. \quad (10)$$

$$\text{Add (9) and (10),} \quad 2z = 12000. \quad (11)$$

$$z = 6000. \quad (12)$$

$$\text{Substitute (6) and (12) in (1),} \quad x = 4000. \quad (13)$$

Hence, \$4000, \$5000, and \$6000 were the sums invested.

Solve the following problems, using three unknowns:

1. The sum of three angles, A , B , and C of a triangle is 180 degrees. $\frac{1}{2}$ of A + $\frac{1}{4}$ of B + $\frac{1}{6}$ of C is 48 degrees, while $\frac{1}{3}$ of A + $\frac{1}{6}$ of B + $\frac{1}{4}$ of C is 30 degrees. How many degrees in each angle?

2. The combined weight of 1 cubic foot each of compact limestone, granite, and marble is 535 pounds. 1 cubic foot of limestone, 2 of granite, and 3 of marble weigh together 1041 pounds, while 1 cubic foot of limestone and 1 of granite together weigh 195 pounds more than 1 cubic foot of marble. Find the weight per cubic foot of each kind of stone.

3. A number is composed of 3 digits whose sum is 7. If the digits in tens' and hundreds' places are interchanged, the number is increased by 180; and if the order of the digits is reversed, the number is decreased by 99. What is the number?

4. The sum of the angles A , B , and C of a triangle is 180 degrees. If B is subtracted from C , the remainder is $\frac{1}{2}$ of A , and when C is subtracted from twice A , the remainder is 4 times B . How many degrees in each angle?

5. The sum of the three sides a , b , c of a certain triangle is 35, and twice a is 5 less than the sum of b and c , and twice c is 4 more than the sum of a and b . What is the length of each side?

6. If x is the number of seconds in the Eastern intercollegiate record for a mile run, y the number in the Western intercollegiate record, and z the number in the world's record, then

$$\begin{cases} x + y + z = 768.95, \\ -x + 2y + z = 518.95, \\ 2x - y + z = 502.75. \end{cases}$$

7. If x is the number of seconds in the Eastern intercollegiate record for a half mile run, y the number in the Western intercollegiate record, and z the number in the world's record, then

$$\begin{cases} 2x + 3y + z = 692.9, \\ 3x + 2y + 2z = 804.6, \\ 2x - y + z = 226.5. \end{cases}$$

8. If x = number of seconds in the world's mile trotting record in 1806, y = number of seconds in the world's record in

1885, and z = number of seconds in the world's record in 1911, then

$$\begin{cases} x + y + z = 426.25, \\ 2x + 4y + 6z = 1584, \\ -x + y + 2z = 186.75. \end{cases}$$

9. Diophantus of Alexandria (see page 19) gives the following problem: "Find three numbers such that the sum of each pair is a given number." Solve this problem if the given numbers are 20, 30, 40 (the numbers actually given by Diophantus).

Diophantus remarks that half the sum of the three given numbers must be greater than each singly. This is to prevent negative numbers in the results.

10. Solve the preceding problem when the given numbers are a , b , c . That is, solve the system $x + y = a$,

$$y + z = b,$$

$$z + x = c.$$

It is interesting to note that Diophantus states his problem in words in its general form but he solves it for a special case, viz. for $x + y = 20$, $y + z = 30$, $z + x = 40$. The Greeks did not use letters to represent numbers in general. Hence they had no formulas such as we now have.

REVIEW QUESTIONS

11. Describe elimination by the process of substitution; also by the process of addition or subtraction. Under what conditions is one or the other of these methods preferable?

12. Describe the solution of a system of three linear equations in three unknowns. Is it immaterial which of the three variables is eliminated first?

13. Can you find a definite solution for two equations each containing three unknowns? Illustrate this by means of the equations $4x - 3y - z = 5$ and $x + y + z = 2$.

DRILL EXERCISES

1. $(1-3x)^2 + (2x+1)^2 = 5x^2 + (2x+6)(4x+27)$.
2. $(1+2x)(2-3x) + (x-4)(x+4) = (x+14)(18-5x) - 1$.
3. $2(x+5)(x-5) = (x-5)(x+1) + (x-2)^2 - 1$.
4.
$$\begin{cases} \frac{x-1}{2} + \frac{y+2}{3} = 3\frac{1}{6}, \\ \frac{x+1}{3} - \frac{y-1}{4} = \frac{7}{12} \end{cases}$$
5.
$$\begin{cases} \frac{x-a}{2} + \frac{y-b}{3} = 1, \\ \frac{x+a}{3} + \frac{y+b}{2} = 1. \end{cases}$$
6.
$$\begin{cases} 2ax - 3by = c, \\ 2ax + 3by = d. \end{cases}$$
7.
$$\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \\ 3x + 2y - 2z = 1. \end{cases}$$
8.
$$\begin{cases} x - y + z = 4, \\ x - 2y + 4z = 6, \\ 2x + y - 3z = 10. \end{cases}$$
9.
$$\begin{cases} x + y + z = 0, \\ 2x - 4y + z = -3, \\ 3x + 2y + 4z = 3. \end{cases}$$

Solve the following equations for x :

10. $(x-a)(x-a) = -x(a-x) + (a+b)x$.
11. $\frac{a}{b} = \frac{x}{c-x}$.
12. $\frac{4a}{3x} + \frac{4a}{x} = \frac{3}{2} + \frac{5a}{6x}$.
13. $\frac{2a-3b}{3x} + \frac{b}{x} + \frac{3a}{2x} = \frac{13a}{6}$.
14. Express the average of the numbers 3, 8, -9, 12; also of the numbers $3a$, b , $2c$, $-5b$.
15. Express the sum of the squares of four consecutive even integers of which $2n$ is the smallest.
16. Express the sum of the squares of four consecutive odd integers of which $2n+1$ is the greatest.
17. A wall is l feet long and h feet high. There are three windows in it k feet wide and m feet high. By how much does the area of the wall exceed that of the three windows? Express the result in terms of l , h , k , and m .

PART TWO

CHAPTER IX

PRODUCTS, QUOTIENTS, AND FACTORS

The principles and processes studied in Part One enabled us to *solve equations* of the first degree involving one or more unknowns, and to *solve all problems* which we could translate into such equations.

In Part Two we begin with factoring which will lead to the *solution of equations* of the second degree and thereby to the *solution of problems* still more difficult.

128. Repeated Factors. Number expressions containing repeated factors have already been considered in Chapter IV. $x \cdot x$ was written x^2 and called *the square of x* , or *x square*; similarly, $(a + b)(a + b)$ was written $(a + b)^2$ and read *the square of the binomial $a + b$* or *$a + b$ squared*.

129. Definitions. Any number written over and to the right of a number expression is called an **index** or **exponent**. If an exponent is a *positive integer*, it shows how many times the expression is to be taken as a factor.

A product consisting entirely of equal factors is called a **power** of the repeated factor. The repeated factor is called the **base** of the power.

E.g. x^3 means $x \cdot x \cdot x$ and is read *the third power of x* or *x cube*; x^5 means $x \cdot x \cdot x \cdot x \cdot x$, and is read *the fifth power of x* or briefly *x fifth*. $(x - y)^3 = (x - y)(x - y)(x - y)$ and is read *$x - y$ cubed* or *the cube of the binomial $x - y$* .

The first power of x is written without an exponent. Thus x means x^1 ; 2 means 2^1 etc.

130. Notice the difference between a *coefficient* and an *exponent*. A coefficient is a *factor*, while an exponent shows how many times some number is used as a factor.

E.g. $5a = a + a + a + a + a$, while $a^5 = a \cdot a \cdot a \cdot a \cdot a$.

EXERCISES

Perform the following indicated multiplications :

- | | | |
|----------------------|---------------------------|-----------------------|
| 1. $2^3, 2^4, 2^5$. | 5. $6^2, 6^3$. | 9. $(a + b + c)^2$. |
| 2. $3^2, 3^3, 3^4$. | 6. $(a + b)^2$. | 10. $(a + b - c)^2$. |
| 3. $4^2, 4^3, 4^4$. | 7. $(c - d)^2$. | 11. $(3 - a)^2$. |
| 4. $5^2, 5^3, 5^4$. | 8. $98^2 = (100 - 2)^2$. | 12. $(3 - b - c)^2$. |

PRODUCT OF TWO POWERS OF THE SAME BASE

131. In the case of factors expressed in Arabic figures multiplications like the following may be carried out in either of two ways.

E.g. $3^2 \cdot 3^4 = 9 \cdot 81 = 729$,

or $3^2 \cdot 3^4 = (3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3) = 3^{2+4} = 3^6 = 729$.

But with literal factors the second process only is possible.

E.g. $a^2 \cdot a^4 = (a \cdot a)(a \cdot a \cdot a \cdot a) = a^{2+4} = a^6$.

EXERCISES

In the following exercises carry out each indicated multiplication in two ways in case this is possible :

- | | | | |
|----------------------|----------------------|--------------------------------|-----------------------------------------|
| 1. $5 \cdot 5^2$. | 4. $7 \cdot 7^3$. | 7. $x^7 \cdot x^4$. | 10. $2^3 \cdot 2^3 \cdot 2^4$. |
| 2. $5^2 \cdot 5^3$. | 5. $a^2 \cdot a^3$. | 8. $t^3 \cdot t^4$. | 11. $3 \cdot 3^3 \cdot 3^3$. |
| 3. $6^2 \cdot 6^2$. | 6. $x^3 \cdot x^2$. | 9. $t^2 \cdot t^3 \cdot t^4$. | 12. $2^2 \cdot 2^3 \cdot 2^2 \cdot 2$. |

132. **Illustrative Problem.** To multiply 2^k by 2^n , k and n being any two positive integers.

Solution. 2^k means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to k factors,

and 2^n means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to n factors,

Hence, $2^k \cdot 2^n = (2 \cdot 2 \cdot 2 \dots \text{to } k \text{ factors}) (2 \cdot 2 \dots \text{to } n \text{ factors})$

$= 2 \cdot 2 \cdot 2 \cdot 2 \dots \text{to } k + n \text{ factors in all.}$

That is,

$$2^k \cdot 2^n = 2^{k+n}$$

The preceding examples illustrate

Principle XVI

133. Rule. *The product of two powers of the same base is found by adding the exponents of the factors and making this sum the exponent of the common base.*

Exponents are added in multiplication only when they apply to the same base.

E.g. $2^8 \cdot 3^2 = 8 \cdot 9 = 72$ cannot be found by adding the exponents.

EXERCISES

Perform the following indicated multiplications by means of Principle XVI:

- | | | |
|----------------------|----------------------------|------------------------------|
| 1. $2^8 \cdot 2^7$. | 5. $3^4 \cdot 3^a$. | 9. $5^{2+a} \cdot 5^{2-a}$. |
| 2. $a^3 \cdot a^7$. | 6. $x^2 \cdot x^a$. | 10. $a^m \cdot a^n$. |
| 3. $3^4 \cdot 3^5$. | 7. $4^a \cdot 4^b$. | 11. $c^x \cdot c^{2-x}$. |
| 4. $x^4 \cdot x^5$. | 8. $3^{2a} \cdot 3^{2b}$. | 12. $x^{ac} \cdot x^{bc}$. |

Perform the following multiplications by means of Principles II and XVI.

- | | |
|----------------------------------------|--------------------------------|
| 13. $2^3(2^2 + 2^4)$. | 16. $a^2(a^3b - ab^3)$. |
| 14. $2^2(3 \cdot 2^4 + 5 \cdot 2^5)$. | 17. $x^{2a}(4x^a + 3x^{3a})$. |
| 15. $4^4(3 \cdot 4^5 - 5 \cdot 4^3)$. | 18. $r^a(5r^{2a} - 3r^a)$. |

Multiply the following:

- | | |
|--------------------------------------------------|-----------------------------------------------|
| 19. $a^2m^2 - b^2m^3$ by m^4 . | 23. $3y^2 + 4y^4 - y^3$ by y . |
| 20. $4 \cdot 3^3 - 5 \cdot 7 \cdot 2$ by 2^4 . | 24. $7x^4 - 5x^3 - 2x$ by x^3 . |
| 21. $4x^2 - 3x^3 + 6x^4$ by x^2 . | 25. $3a^2b^4 + 2a^2b - 4a^2b^2$ by a^3 . |
| 22. $5x - 3x^2 + 2x^4$ by x^4 . | 26. $4a^{2b} - a^{3c} + a^{2d}$ by a^{2b} . |

PRODUCTS OF MONOMIALS

134. In multiplying together monomials like $3a^2bc$ and $2ab^2cd$ it is convenient to arrange the factors so that the same letters are associated together and likewise the numerical coefficients. This we are permitted to do by Principle XIII.

$$\text{Thus, } 3a^2bc \times 2ab^2cd = (3 \cdot 2)(a^2 \cdot a)(b \cdot b^2)(c \cdot c)d = 6a^3b^3c^2d.$$

135. Notice that in the product *the exponent of each letter is the sum of the exponents of this letter in the factors and the numerical coefficient is the product of the numerical coefficients of the factors.*

$$\text{E.g. } (2a^2b^2)(5a^4b^3c) = 10a^{4+2}b^{2+3}c = 10a^6b^5c.$$

This is a convenient rule for finding the product of two or more monomials.

Multiply

EXERCISES

1. $3ab$ by $5a^2b^2$.
2. $4x^3$ by $3xy^2$.
3. $2xyz$ by $3x^2yz$.
4. $6k^3$ by $7ak^2$.
5. $3x^4y^2$ by $4xy^3$.
6. $5a^2b^3c$ by ab^4c .
7. $2x^3b^4c^2$ by $5xb^4c$.
8. $3a^3b^4c$ by abd^4 .
9. $2x(4 + 7x^4 - 3x^2y)$.
10. $4yx(3y^2x^3 - y^3x^2 + y^4x^4)$.
11. $5a^2b^2(a^3 - b^3 + a^2b^2)$.
12. $4x^4y^3(3ax - 4by + 2xy)$.
13. $(a+b)^2(a-b)$.
14. $(a-b)^2(a+b)$.
15. $(a+b)^2(a-b)^2$.
16. $(a^3 + a^2b + ab^2 + b^3)(a-b)$.
17. $(a+b-c)(a+b+c)$.
18. $(3x-2y-1)(2x+y)$.
19. $(1+a+a^2)(1-a)$.
20. $(1-a+a^2-a^3)(1+a)$.
21. $(a+b)(a-b)(a^2+b^2)$.
22. $(a+b)(a^2-ab+b^2)$.
23. $(a-b)(a^2+ab+b^2)$.
24. $(x^2+x+1)(x^2-x+1)$.
25. $(x+y)(x^2-x^2y+xy^2-y^3)$.
26. $(x-y)(x^3+x^2y+xy^2+y^3)$.
27. $(x^2-xy+y^2)(x^2+xy+y^2)$.
28. $(x^2+2xy+y^2)(x-y)^2$.
29. $(x^2-y^2)(x^4+x^2y^2+y^4)$.
30. $(x^3+y^3)(x^4-x^2y^2+y^4)$.
31. $(x^2+y^2)(x+y)(x-y)$.
32. $(x-y)(x^2+xy+y^2)(x^3+y^3)$.

QUOTIENT OF TWO POWERS OF THE SAME BASE

136. **Illustrative Problem.** To divide x^6 by x^4 .

Since by § 65 the quotient times the divisor equals the dividend, we seek an expression which multiplied by x^4 equals x^6 .

Since by Principle XVI two powers of the same base are multiplied by adding their exponents, the expression sought must be that power of x whose exponent added to 4 equals 6. Hence the exponent of the quotient is $6 - 4 = 2$. That is, $x^6 \div x^4 = x^{6-4} = x^2$.

EXERCISES

Perform the following indicated divisions:

- | | | |
|------------------------|---------------------------|------------------------------------|
| 1. $2^4 \div 2^2$. | 8. $5^{13} \div 5^{12}$. | 15. $x^4 \div x^3$. |
| 2. $2^3 \div 2^2$. | 9. $7^{24} \div 7^{22}$. | 16. $t^{14} \div t^4$. |
| 3. $2^4 \div 2$. | 10. $8^3 \div 8$. | 17. $m^3 \div m$. |
| 4. $3^3 \div 3^2$. | 11. $6^4 \div 6^2$. | 18. $n^6 \div n^2$. |
| 5. $3^4 \div 3$. | 12. $a^3 \div a^2$. | 19. $(20)^4 \div (20)$. |
| 6. $3^4 \div 3^2$. | 13. $a^4 \div a^3$. | 20. $(101)^{14} \div (101)^{13}$. |
| 7. $9^{11} \div 9^2$. | 14. $m^4 \div m^2$. | 21. $(41)^7 \div (41)^6$. |

The preceding exercises illustrate

Principle XVII

137. **Rule.** *The quotient of two powers of the same base is a power of that base whose exponent is the exponent of the dividend minus that of the divisor.*

For the present only those cases are considered in which the exponent of the dividend is greater than or equal to that of the divisor.

Notice that Principle XVII does not apply to powers of different bases.

E.g. $3^7 \div 2^4$ does not equal any integral base to the power, $7 - 4$. This division can be performed only by first multiplying out both dividend and divisor.

EXERCISES

Perform the following indicated divisions by means of Principle XVII:

- | | | |
|---------------------------|------------------------------|----------------------------------|
| 1. $2^7 \div 2^3$. | 6. $x^{4a} \div x^{2a}$. | 11. $x^{2a+b} \div x^{a+b}$. |
| 2. $a^7 \div a^3$. | 7. $3^{2a-1} \div 3^{a-2}$. | 12. $w^{2a} \div w^a$. |
| 3. $3^4 \div 3^2$. | 8. $5^{a+b} \div 5^{a+2}$. | 13. $(17)^{14} \div (17)^{13}$. |
| 4. $x^4 \div x^2$. | 9. $x^{a+4} \div x^{a+2}$. | 14. $5x^6 \div 2x^2$. |
| 5. $3^{3a} \div 3^{2a}$. | 10. $t^{4a} \div t^a$. | 15. $(12)^4 \div (12)^3$. |

In the following use Principles III, V, and XVII:

- | | |
|----------------------------------------------|-------------------------------------------------------|
| 16. $(3 \cdot 2^4 + 5 \cdot 2^3) \div 2^2$. | 21. $(4x^3 - 5x^2 + x^4) \div x^2$. |
| 17. $(3 \cdot 4^3 - 5 \cdot 4^4) \div 4^2$. | 22. $(3a^3 + 9a^4 - 2a^5) \div a^2$. |
| 18. $(a^3b - a^4b^2) \div a^2$. | 23. $(12x^2y - 11x^2y^2 + 5x^4) \div x^2$. |
| 19. $(4x^3 + 3x^4) \div x^2$. | 24. $(x^{2m+4} + x^{2m+3} - 5x^{m+2}) \div x^{m+1}$. |
| 20. $(a^2m^4 - b^2m^3) \div m^3$. | 25. $(y^{3a+b} + y^{2a-b} - y^a) \div y^a$. |

138. The process of division by subtracting exponents leads in certain cases to strange results.

Thus, according to this process, $x^4 \div x^4 = x^{4-4} = x^0$, which is as yet without meaning, since an exponent has been defined only when it is a *positive integer*. The exponent zero cannot indicate, as in the case of a positive integer, how many times the base is used as a factor. We know, however, that $x^4 \div x^4 = 1$, since any number divided by itself equals unity. Hence if we use the symbol x^0 it must be interpreted to mean 1, no matter what number x represents.

Again by this process, $x^2 \div x^4 = x^{2-4} = x^{-2}$, which is as yet without meaning, since negative exponents have not been defined. But we know that $x^2 \div x^4 = \frac{x^2}{x^4} = \frac{1}{x^2}$ by Principle XV. Hence if we use the symbol x^{-2} , it must be interpreted to mean $\frac{1}{x^2}$. Negative exponents and also fractional exponents are considered in detail in the Advanced Course.

DIVISION OF MONOMIALS

139. In dealing with the quotient of two monomials the indicated division may be written in the form of a fraction and the factors common to dividend and divisor may be cancelled by Principle XV as in the following examples:

$$(1) \quad 15 a^3 b^2 c + 3 a^2 b x^2 y = \frac{15 a^3 b^2 c}{3 a^2 b x^2 y} = \frac{5 abc}{x^2 y}.$$

$$(2) \quad 12 a^2 x + 8 ax = \frac{12 a^2 x}{3 ax} = \frac{4 a}{1} = 4 a.$$

Divide:

EXERCISES

1. $4 \cdot 7 \cdot 9$ by $2 \cdot 3$.
2. $12 \cdot 8 \cdot 20$ by $2 \cdot 4 \cdot 5$.
3. $6 x^2 y^2 z$ by $2 xyz$.
4. $6^4 \cdot 3^4 \cdot x^2$ by $6^2 \cdot 5^2 \cdot x^2$.
5. $12 x^{12} y^{13}$ by $4 xy^2 z$.
6. $5 a^4 b^{11} c$ by $ab^4 c^2$.
7. $10 x^4 b^4 c^3$ by $2 x b^4 c$.
8. $36 x^4 y^3$ by $6 x^2 y^5$.
9. $4 x^2 y^3 - 3 x^3 y^2$ by $x^2 y^2$.
10. $18 x^4 y^4 - 12 x^3 y^3 + 6 x^2 y^2$ by $6 x^2 y^2$.
11. $49 a^4 + 21 a^3 - 7 a$ by $7 a$.
12. $12 ax^4 y^3 - 16 a^2 x^2 y^2 + 8 a^3 xy$ by $4 axy$.
13. $2 x^{3a} + 4 x^{4a} - 8 x^{2a}$ by $2 x^a$.
14. $6 x^{2n+1} + 12 x^{3n+1} - 10 x^{n+1}$ by $2 x^{n+1}$.
15. $4 x^{13} - 6 x^{11} b - 10 x^4 c$ by $2 x^4$.
16. $10 a^3 b^3 - a^2 b^3 + 15 a^4 b^4$ by $5 a^2 b^2$.

FACTORS OF NUMBER EXPRESSIONS

140. Factors are of great importance in arithmetic. For instance, the multiplication table consists of pairs of factors whose products are committed to memory for constant use. Likewise in algebra the factors of certain special forms of number expressions are so important that they must be known at sight.

Definition. An expression containing no fractions is said to be **prime** if it has no factors in the integral form except itself and 1.

Thus 2, 3, x , $x + 2$, $a^2 + b^2$, are prime expressions.

MONOMIAL FACTORS

(141.) If the terms of a polynomial contain a **common monomial factor**, the polynomial may be divided by the monomial, and the quotient and the divisor are the factors of the polynomial.

E.g. $a^2 - ab = a(a - b)$, $5xy - 3x^2y + 4x^2y = xy(5 - 3x + 4x^2)$, and $6x^{2a}y^{4b} + 8x^{4a+3}y^{6b} - 12x^{5a}y^{8b} = 2x^{2a}y^{4b}(3 + 4x^{2a+3}y^{2b} - 6x^{3a}y^{4b})$.

Observe that factoring the various terms of a polynomial does not factor the polynomial.

E.g. $a^2 + ax + ab + by$ is not factored by writing it
 $a(a + x) + b(a + y)$.

Likewise $10a^2bc - 15ab^2c + 20abc^2$ is not factored, although *each term* is in the factored form. But if $10a^2bc - 15ab^2c + 20abc^2$ is written in the form $5abc(2a - 3b + 4c)$, it is then factored.

Note that removing a common factor from the terms of a polynomial is nothing more than the application of Principle I.

EXERCISES

Factor the following polynomials:

1. $7a^2 + 14ab + 21a$.
2. $13a^4b - 16a^3b^4 - 2a^3b^2$.
3. $15xy^4 - 20x^2y + x^2y^2$.
4. $9v^3w^4 + 21v^4w^3 - 18v^2w^2$.
5. $12a^4b^3 - 8a^3b^4 - 6a^2b^2$.
6. $11a^4x^2 - 44a^3x^4 + 33ax$.
7. $72x^4b^3a - 36x^4b^2a^2 - 48x^3b^3a^4$.
8. $84x^2b^7y^4 + 18x^3b^5y^4 + 12x^2b^3y^2$.
9. $17a^4b^3c^2 + 51a^3b^4c^3 - 34a^2b^2c^4$.
10. $38a^{12}b^{14}c^4 - 76a^{11}b^{12}c^3 - 76a^{13}b^{11}c^2$.
11. $4x^{2a}y^{3b} + 6x^{3a}y^{2b} - 8x^{5a}y^{4b}$.
12. $3a^{4n+2}b^{3n+4} + 6a^{6n+4}b^{3n+3} - 12a^{5n+3}b^{5n+6}$.

TRINOMIAL SQUARES

142. In §§ 89 and 90 we found by multiplication:

$$(a + b)^2 = a^2 + 2ab + b^2, \quad (1)$$

and
$$(a - b)^2 = a^2 - 2ab + b^2. \quad (2)$$

By means of these formulas we may square the sum or difference of any two number expressions.

E.g. $(3x + 7)^2 = (3x)^2 + 2(3x) \cdot 7 + 7^2 = 9x^2 + 42x + 49.$

$$(a + b + c)^2 = [(a + b) + c]^2 = (a + b)^2 + 2(a + b)c + c^2.$$

$$[(5 + r) - (s - t)]^2 = (5 + r)^2 - 2(5 + r)(s - t) + (s - t)^2.$$

The last expressions may now be reduced by performing the indicated operations.

In this manner write the squares of the following binomials. Read as many as possible at sight.

- | | | | |
|-----------------------------|------------------------------|-----------------|-------------------|
| 1. $t + 8.$ | 3. $3x - 5y.$ | 5. $m^5 + 3n.$ | 7. $3y^3 - 2z^2.$ |
| 2. $r - 12.$ | 4. $6a - 7.$ | 6. $7x^2 + 3y.$ | 8. $8m^2 - 7n^2.$ |
| 9. $5a^3 - 3b^3.$ | 12. $(3a - 2b) + 5.$ | | |
| 10. $(a - 3b) - 2(b + c).$ | 13. $7x - (4y - s).$ | | |
| 11. $(x - y) + 2(x + y^2).$ | 14. $(m^3 - 3) + (m^3 + n).$ | | |

143. The binomial $a + b$ is one of the two equal factors of $a^2 + 2ab + b^2$, and is called the **square root** of this trinomial.

Likewise $a - b$ is the square root of $a^2 - 2ab + b^2$.

In each case a is the square root of a^2 and b of b^2 . Hence $2ab$ is twice the product of the square roots of a^2 and b^2 .

From the squares obtained in the last article, we learn to distinguish whether any given trinomial is a perfect square, as in the following examples:

1. $x^2 + 4x + 4$ is in the form of (1), since x^2 and 4 are squares each with the sign +, and $4x$ is twice the product of the square roots of x^2 and 4. Hence

$$x^2 + 4x + 4 = x^2 + 2(2x) + 2^2 = (x + 2)(x + 2) = (x + 2)^2.$$

2. $x^2 - 4x + 4$ is in the form of (2), since it differs from (1) only in the sign of the middle term. Thus

$$x^2 - 4x + 4 = x^2 - 2(2x) + 2^2 = (x - 2)(x - 2) = (x - 2)^2.$$

3. $16x^2 + 9y^2 + 12xy$ is not in the form (1) since $12xy$ is not equal to $2(4x)(3y)$.

Definition. A trinomial which is the product of two equal factors is called a **trinomial square**.

EXERCISES

1. State the facts concerning a trinomial which make it a perfect square.

Determine whether the following are trinomial squares, and if so indicate the two equal factors.

- | | | |
|------------------------------|--------------------------------|-----------------------------|
| 2. $x^2 + 2xy + y^2$. | 6. $m^2 + n^2 - 2mn$. | 10. $a^3 + b^3 - 2a^2b^4$. |
| 3. $x^2 - 2xy + y^2$. | 7. $r^2 + s^2 + 2rs$. | 11. $64 + t^2 - 16t$. |
| 4. $x^4 + 2x^2y^2 + y^4$. | 8. $4x^2 - 8xy + 4y^2$. | 12. $16 + x^2 - 8x$. |
| 5. $x^4 - 2x^2y^2 + y^4$. | 9. $a^6 + b^6 + 2a^3b^3$. | 13. $9 - 6y + y^2$. |
| 14. $25x^2 + 16y^2 + 40xy$. | 18. $16a^2 + 25b^2 - 50ab$. | |
| 15. $4m^2 + n^2 + 2mn$. | 19. $16a^2 + 25b^2 + 40ab$. | |
| 16. $100 + s^2 + 20s$. | 20. $81 - 270b + 225b^2$. | |
| 17. $64 + 49 + 112$. | 21. $121 + 4a^2b^4 - 22ab^2$. | |

144. From the foregoing examples we see that a *trinomial* is a perfect square if it contains two terms which are squares each with the sign +, while the third term, whose sign is either + or -, is twice the product of the square roots of the other two. Then the square root of the trinomial is the sum or the difference of these square roots according as the sign of the third term is + or -.

Since on multiplying we find $(a - b)^2$ and $(b - a)^2$ give the same result, we may write the factors of $a^2 - 2ab + b^2$ either $(a - b)(a - b)$ or $(b - a)(b - a)$. That is, the square root of $(a - b)^2$ is either $a - b$ or $-(a - b)$. Likewise the square root of a^2 is either $+a$ or $-a$. See page 176.

EXERCISES

Factor the following. If any one of the trinomials is found not to be a square, show where it is lacking.

- | | |
|-----------------------------------|----------------------------------|
| 1. $9 + 2 \cdot 3 \cdot 4 + 16$. | 16. $121 + 4x^2 - 44x^4$. |
| 2. $x^2 + 4y^2 + 4xy$. | 17. $16x^4 + 64y^4 - 64x^2y^2$. |
| 3. $9x^2 + 18xy + 9y^2$. | + 18. $81a^2 - 216a + 144$. |
| 4. $4x^2 + 4xy + y^2$. | 19. $4a^2 + 8ab^2 + 4b^3$. |
| 5. $4x^2 + 8xy + 4y^2$. | 20. $9b^4 + 18b^2c^4 + 9c^8$. |
| 6. $25x^2 + 12xy + 4y^2$. | 21. $4x^2 + 4y^2 - 8xy$. |
| 7. $16x^2 + 16xy + 4y^2$. | 22. $9a^2 - 16ab + 4b^2$. |
| 8. $9r^2 + 36rs + 25s^2$. | 23. $9x^4 - 24a^2b + 16b^2$. |
| 9. $16x^2 + 8x^2y + y^2$. | 24. $25 + 49x^2 - 70x$. |
| 10. $4x^2 + 12x^2a^2 + 9a^4$. | 25. $-30ab^2 + 9a^3 + 25b^4$. |
| 11. $a^{10} + 6a^5b + 9b^2$. | 26. $16a^2 - 24ab + 9b^2$. |
| 12. $(a+1)^2 + 2(a+1)b + b^2$. | 27. $36x^2 - 84x + 49$. |
| 13. $(x+3)^2 + 4(x+3)y + 4y^2$. | + 28. $25 - 90 + 81$. |
| 14. $x^2 + 12x^2 + 36$. | 29. $64x^2 - 32x + 9$. |
| 15. $a^4 + 18a^2 + 12$. | X 30. $(3+a)^2 + b^2 - 2b(3+a)$ |

THE DIFFERENCE OF TWO SQUARES

145. In § 91 we found by multiplication,

$$(a+b)(a-b) = a^2 - b^2.$$

By means of this formula the product of the sum and difference of any two number expressions may be found.

$$\begin{aligned} \text{E.g. } (x+y-z)(x+y+z) &= [(x+y)-z][(x+y)+z] \\ &= (x+y)^2 - z^2. \end{aligned}$$

In this manner form the following products. Verify the first ten by actual multiplication. Read at sight the first seven.

- | | |
|-----------------------------------|---------------------------|
| 1. $(4a+5b)(4a-5b)$. | 4. $(5-6t^2)(5+6t^2)$. |
| 2. $(24x+12y)(24x-12y)$. | 5. $(3x-2y)(3x+2y)$. |
| 3. $(16a^2b^2-3c)(16a^2b^2+3c)$. | 6. $(x^2-y^2)(x^2+y^2)$. |

7. $(3x^4 - 5y^4)(3x^4 + 5y^4)$.
8. $[x + (y - z)][x - (y - z)]$.
9. $(x^m + y^n)(x^m - y^n)$.
10. $[c - (a - b)][c + (a - b)]$.
11. $[x - (y + z)][x + (y + z)]$.
12. $(a + b + c)(a - b - c)$.
13. $(a + b - c)(a - b + c)$.
14. $(a - b + c)(a - b - c)$.
15. $(r - y - z)(r - y + z)$.
16. $(a + b + c)(a + b - c)$.

146. From the preceding examples we see that *every binomial which is the difference between two perfect squares is composed of two binomial factors; namely, the sum and the difference of the square roots of these squares.*

E.g. $16x^2 - 9y^2$ is the difference between two squares $(4x)^2$ and $(3y)^2$. Hence we have

$$16x^2 - 9y^2 = (4x)^2 - (3y)^2 = (4x + 3y)(4x - 3y).$$

$$\begin{aligned}\text{Again, } (a - 3b)^2 - (2a + b)^2 &= [(a - 3b) + (2a + b)][(a - 3b) - (2a + b)] \\ &= (a - 3b + 2a + b)(a - 3b - 2a - b) \\ &= (3a - 2b)(-a - 4b).\end{aligned}$$

EXERCISES

Factor each of the following. Read as many as possible at sight.

- | | | |
|---------------------|-------------------------|------------------------|
| 1. $x^2 - 4y^2$. | 8. $a^2 - 1$. | 15. $4 - (x - 2y)^2$. |
| 2. $9x^2 - 36y^2$. | 9. $1 - 9x^4$. | 16. $16a - 25ab^2$. |
| 3. $x^4 - b^2$. | 10. $4 - 36a^2$. | 17. $49x - 4xy^2$. |
| 4. $4x^3 - 9b^3$. | 11. $1 - 64a^3$. | 18. $225 - 64x^2y^4$. |
| 5. $16a^4 - 9b^4$. | 12. $144x^2b^4 - 1$. | 19. $576a - 144ay^2$. |
| 6. $64 - b^2$. | 13. $256a^4b^6 - c^2$. | 20. $5^3 - 3^3$. |
| 7. $1 - b^2$. | 14. $1 - (x + y)^2$. | 21. $x^4 - 81y^2$. |

147. It is important to determine whether a given expression can be written as the difference of two squares.

$$\begin{aligned} \text{E.g. } a^2 + b^2 + 2ab - c^2 &= (a+b)^2 - c^2 = (a+b+c)(a+b-c). \\ \text{Also, } c^2 - a^2 + 2ab - b^2 &= c^2 - (a^2 - 2ab + b^2) = c^2 - (a-b)^2 \\ &= (c-a+b)(c+a-b). \end{aligned}$$

EXERCISES

In the following determine whether each is the difference of two squares, and if so, factor it accordingly.

1. $x^2 - (y-z)^2$.
2. $(x-y)^2 - z^2$.
3. $a^2 + b^2 - 2ab - 4$.
4. $x^2 + y^2 - 2xy - z^2$.
5. $4a^2b^2 - (a^2 + b^2 - c^2)^2$.
6. $a^3 - (b^2 + c^2 + 2bc)$.
7. $(2a-5)^2 - (3a+1)^2$.
8. $(3x^2 - y)^2 - (x+y)^2$.
9. $(3a-2b)^2 - (8a+5b)^2$.
10. $(3m-4)^2 - (2m+3)^2$.
11. $(2r+s)^2 - (3r-s)^2$.
12. $81 - (a+b+c)^2$.
13. $x^4 + x^2 + 1 - 4a^2$.
14. $a^2 - (x+2y)^2$.
15. $9x^2 - (a-b)^2$.
16. $25m^2 - (3r+2s)^2$.
17. $4c^2 - (4a^2 + 12ab - 9b^2)$.
18. $9a^2 + 6ab + b^2 - c^2$.
19. $16x^2 - a^2 + 4ab - 4b^2$.
20. $a^2 - 2ab + b^2 - c^2$.
21. $c^2 - (a^2 + 2ab + b^2)$.
22. $c^2 - (a^2 - 2ab + b^2)$.
23. $(a+b)^2 - (a-b)^2$.
24. $a^2 + 4ab + b^2 - x^2$.
25. $a^2 + 4ab + 4b^2 - x^2$.
26. $9a^2 + 16b^2 - 32ab - x^4$.

DRILL EXERCISES

1. $16c - (41 - 7c) + (15 - 8c)$.
2. $-(5a - 3c) - (2c - 8a) + 3a$.
3. $-(-12x - 7y - 15x) - (-9y + 8x + 3y)$.
4. $(19x + 4y - 32x - 17x) - 12x - (49y + 18x - 70x)$.
5. $17a - 3 - (7a - 2) + (6a - 5)$.
6. $7(m + 6) + 10m = 42 - 8(2m + 2) + 181$.
7. $20 - 3(x - 4) + 2x = 2x + 17$.
8. $\frac{x}{x-2} + \frac{4x}{x+3} = 5$.
9. $\frac{x+a}{a+b} + \frac{x+b}{a-b} = 1$.
10. $\begin{cases} x + 2y = 4, \\ 2x + y = -1. \end{cases}$
11. $\begin{cases} 5x + 9y = 19, \\ 3x - y = 5. \end{cases}$
12. $\begin{cases} 3x - 7y = -11, \\ 2x + y = 4. \end{cases}$
13. $\begin{cases} 5x - 3y = 4 - 2x + 7y, \\ 5y + x = 7. \end{cases}$
14. $\begin{cases} x - 3y + z = 10, \\ 2x + y - z = 1, \\ 3x - 2y + 5z = 31. \end{cases}$
15. $\begin{cases} x + y + z = 1, \\ 3x + 4y - z = 1, \\ -2x - y + 3z = 5. \end{cases}$

16. Represent by a graph the progress of a train going 40 miles per hour. Is the distance an increasing or decreasing variable as the time increases?

17. Represent by a graph the equation: $2x - 3y = 5$. Does y increase or decrease as x increases?

18. Graph $x + 2y = 6$ and state whether y increases or decreases as x increases.

19. How do you represent a fraction whose numerator is 3 less than twice that of $\frac{m}{n}$ and whose denominator is equal to 3 times the sum obtained by adding 2 to the denominator of this fraction?

20. There are two numbers such that if one half their product is divided by twice their sum the result is 12 times their difference. Write an equation representing this relation between the numbers.

DIVISION BY A POLYNOMIAL

148. The simplest case of division by a polynomial is that in which the dividend can be resolved into two factors, one being the given polynomial divisor and the other a monomial.

E.g. To divide $4x^3 + 4x^2y$ by $x + y$, factor the dividend, and we have

$$4x^2(x + y) \div (x + y) = 4x^2.$$

In case the dividend cannot be factored in this manner, then, if the division is possible, the quotient must be a polynomial. The process of finding the quotient under such circumstances is best shown by studying a particular case.

Illustrative Example 1. Consider the product

$$(x^2 + 2xy + y^2)(x + y) = x^3(x + y) + 2xy(x + y) + y^2(x + y).$$

The products, $x^2(x + y)$, $2xy(x + y)$, and $y^2(x + y)$ are called **partial products**, and their sum, $x^3 + 3x^2y + 3xy^2 + y^3$, the **complete product**.

In dividing $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$ the quotient must be such a polynomial that when its terms are multiplied by $x + y$ the results are these partial products, which in the solution are called 1st, 2d, and 3d products.

The work may be arranged as follows:

Dividend or product:	$x^3 + 3x^2y + 3xy^2 + y^3$	$x + y$, divisor.
1st product, $x^2(x + y)$:	$x^3 + x^2y$	$x^2 + 2xy + y^2$,
Dividend minus 1st product:	$2x^2y + 3xy^2 + y^3$	[quotient.
2d product, $2xy(x + y)$:	$2x^2y + 2xy^2$	
Dividend minus 1st and 2d products:	$xy^2 + y^3$	
3d product, $y^2(x + y)$:	$xy^2 + y^3$	
Dividend minus 1st, 2d, and 3d products:	0	

Explanation. Since the dividend or product contains the term x^3 , and since one of the factors, the divisor, contains the term x , the other factor, the quotient, must contain the term x^2 . Multiplying

this term of the quotient by the divisor, we obtain the first partial product, $x^3 + x^2y$.

Subtracting the first partial product from the whole product $x^3 + 3x^2y + 3xy^2 + y^3$, the remainder is $2x^2y + 3xy^2 + y^3$, which is the product of the divisor and that part of the quotient which has not yet been found. Since this product contains the term $2x^2y$ and the divisor contains the term x , the quotient must contain the term $2xy$. The product of $2xy$ and $x + y$ is the second partial product.

In like manner the third partial product is $xy^2 + y^3$.

Subtracting the third partial product the remainder is zero. Hence the sum of the three partial products thus obtained is equal to the whole product, and it follows that $x^2 + 2xy + y^2$ is the required quotient.

149. Problems in division may be checked by substituting any convenient values for the letters. For example, in this case, $x = 1$, $y = 1$, reduces the dividend to 8, the divisor to 2, and the quotient to 4, which verifies the correctness of the result.

Since division by zero is impossible (see Advanced Course), care must be taken not to select such values for the letters as will reduce the divisor to zero.

Illustrative Example 2. Divide $2x^4 + x^3 - 7x^2 + 5x - 1$ by $x^2 + 2x - 1$.

Solution.

		[divisor.
Dividend or product:	$2x^4 + x^3 - 7x^2 + 5x - 1$	$x^2 + 2x - 1,$
1st product, $2x^2(x^2 + 2x - 1)$:	$2x^4 + 4x^3 - 2x^2$	$2x^2 - 3x + 1,$
Dividend minus 1st product:	$-3x^3 - 5x^2 + 5x - 1$	[quotient.
2d product, $-3x(x^2 + 2x - 1)$:	$-3x^3 - 6x^2 + 3x$	
Dividend minus 1st and 2d products:	$+x^2 + 2x - 1$	
3d product, $1 \cdot (x^2 + 2x - 1)$:	$x^2 + 2x - 1$	
Dividend minus 1st, 2d, and 3d products:	0	

Check. Substitute $x = 2$ in dividend, divisor, and quotient.

Illustrative Example 3. Divide $20a^2 - 4 + 18a^4 + 18a - 19a^3$ by $2a^2 - 3a + 4$.

Solution. Arranging dividend and divisor according to the descending powers of a , we have

		[divisor.
Dividend or product:	$18a^4 - 19a^3 + 20a^2 + 18a - 4$	$ 2a^2 - 3a + 4,$
1st product:	$18a^4 - 27a^3 + 36a^2$	$ 9a^2 + 4a - 2,$
Dividend minus 1st product:	$8a^3 - 16a^2 + 18a - 4$	[quotient.
2d product:	$8a^3 - 12a^2 + 16a$	
Dividend minus 1st and 2d products:	$-4a^2 + 2a - 4$	
3d product:	$-4a^2 + 6a - 8$	
Dividend minus all products:	$-4a + 4$	

Since $2a^2$ is not contained in $-4a$, the division ends and $-4a + 4$ is the remainder. As in arithmetic we write this as the numerator of a fraction whose denominator is the divisor. Hence, the complete result is $9a^2 + 4a - 2 + \frac{4 - 4a}{2a^2 - 3a + 4}$.

Check. Substitute $a = 1$ in dividend, divisor, and quotient.

150. From a consideration of the preceding examples the process of dividing by a polynomial is described as follows:

1. Arrange the terms of dividend and divisor according to descending (or ascending) powers of some common letter.

2. Divide the first term of the dividend by the first term of the divisor. This quotient is the first term of the quotient.

3. Multiply the first term of the quotient by the divisor and subtract the product from the dividend.

4. Divide the first term of this remainder by the first term of the divisor, obtaining the second term of the quotient. Multiply the divisor by the second term of the quotient and subtract, obtaining a second remainder.

5. Continue in this manner until the last remainder is zero, or until a remainder is found whose first term does not contain as a factor the first term of the divisor. In case no remainder is zero, the division is not exact.

EXERCISES

Check the result in each case, being careful to substitute such numbers for the letters as do not make the divisor zero.

Divide the following:

1. $a^2 + 2ab + b^2$ by $a + b$.
2. $a^2 - 2ab + b^2$ by $a - b$.
3. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
4. $2x^3 + 2x^2y - 4x^2 - x - 4xy - y$ by $x + y$.
5. $x^3 + xy^2 - x^2y - y^3$ by $x - y$.
6. $x^3 + 4x^2 + x - 6$ by $x + 3$.
7. $x^3 + 4x^2 + x - 6$ by $x - 1$.
8. $x^4 - 6x^3 + 2x^2 - 3x + 6$ by $x - 1$.
9. $x^3 + 3x^2y + 3xy^2 + y^3$ by $x^2 + 2xy + y^2$.
- X 10. $x^3 - 8x^2 + 75x - 5$.
11. $2a^3 + 19a^2b + 3ab^2$ by $2a + b$.
12. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x - y$.
13. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ by $x^2 + 2xy + y^2$.
14. $x^4 + x^3y + xy^3 + y^4$ by $x + y$.
15. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.
16. $x^4 - y^4$ by $x - y$.
17. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
18. $2x^4 + 11x^3 - 26x^2 + 14x + 3$ by $x^2 + 7x - 3$.
19. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.
20. $x^5 - x^4 - 27x^3 + 10x^2 - 30x - 200$ by $x^2 - 4x - 10$.
21. $3x^2 - 4xy + 8xz - 4y^2 + 8yz - 3z^2$ by $x - 2y + 3z$.
22. $9r^2s^2 - 4r^2t^2 + 4rst^2 - s^2t^2$ by $3rs - 2rt + st$.
23. $9a^2b^2 + 16x^3 - 4a^2 - 36b^2x^3$ by $3ab + 6bx - 2a - 4x$.
24. $x^3 + x^2y + x^2z - xyz - y^2z - yz^2$ by $x^2 - yz$.
25. $a^5 + a^4b + a^3 - a^3b^2 - 2ab^2 + b^3$ by $a^2 + ab - b^2$.
26. $a^3 + b^3 + 3ab - 1$ by $a + b - 1$.
27. $a^{2k} - 3a^{2k}b^k + 3a^kb^{2k} - b^{3k}$ by $a^k - b^k$.

THE SUM OF TWO CUBES

151. **Example.** Divide $a^3 + b^3$ by $a + b$.

Since the quotient multiplied by the divisor equals the dividend we have, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

This shows that *every binomial which is the sum of two cubes is the product of two factors, one of which is the sum of the numbers, and the other is the sum of their squares minus their product.*

$$\text{E.g. (1)} \quad x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

$$\begin{aligned} \text{(2)} \quad 8a^3 + 27b^3 &= (2a)^3 + (3b)^3 \\ &= (2a + 3b)(4a^2 - 2a \cdot 3b + 9b^2). \end{aligned}$$

$$\text{(3)} \quad x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4).$$

Notice the difference between the trinomial $x^2 - xy + y^2$ and the trinomial square $x^2 - 2xy + y^2$.

EXERCISES

Determine whether each of the following is the sum of two cubes, and if so, find the factors. Check the results by multiplication.

- | | | |
|--------------------|-------------------------|-------------------------|
| 1. $x^3 + y^3$. | 10. $8a^3 + 27b^3$. | 19. $64x^3 + 27y^3$. |
| 2. $a^3 + 8b^3$. | 11. $8u^3 + 64b^3$. | 20. $8^3 + 10^3$. |
| 3. $27a^3 + b^3$. | 12. $w^3x^6 + x^9a^3$. | 21. $1 + 729x^6$. |
| 4. $8a^3 + 1$. | 13. $1 + 8a^3b^3$. | 22. $x^6 + y^{12}$. |
| 5. $1 + 64x^3$. | 14. $64x^3 + 343$. | 23. $a^9 + b^3$. |
| 6. $2^3 + 3^3$. | 15. $1 + a^3$. | 24. $27r^3 + 125s^3$. |
| 7. $125 + 729$. | 16. $a^3 + 9b^3$. | 25. $x^9 + 27y^6$. |
| 8. $1 + 125x^6$. | 17. $125x^3 + y^6$. | 26. $64 + a^9$. |
| 9. $27x^6 + 1$. | 18. $1 + x^3$. | 27. $a^3b^6 + x^3y^9$. |

28. Find whether $x^3 + y^3$ is exactly divisible by $x - y$; by $x^2 - y^2$; by $x^2 + y^2$. What binomial is a divisor of the sum of two cubes? What is the quotient?

THE DIFFERENCE OF TWO CUBES

152. **Example.** Divide $a^3 - b^3$ by $a - b$ and obtain

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

This shows that the *difference of the cubes of two numbers is the product of two factors, one of which is the difference of the numbers, and the other the sum of the squares of the numbers plus their product.*

E.g. (1) $x^3 - y^3 = (x - y)(x^2 + xy + y^2).$

(2) $8a^3 - 64b^3 = (2a)^3 - (4b)^3$
 $= (2a - 4b)(4a^2 + 2a \cdot 4b + 16b^2).$

(3) $a^3 - b^3 = (a^3)^3 - (b^3)^3 = (a^3 - b^3)(a^6 + a^3b^3 + b^6).$

Notice the difference between the factor $x^2 + xy + y^2$, and the trinomial square $x^2 + 2xy + y^2$.

EXERCISES

Determine whether each of the following is the difference of two cubes, and if so, find the factors. Check the results by multiplication.

- | | | |
|---------------------|--------------------|-------------------------|
| 1. $r^3 - s^3.$ | 9. $27 - 125 a^3.$ | 17. $8 x^4 - y^3.$ |
| 2. $8 a^3 - b^3.$ | 10. $x^6 - y^6.$ | 18. $64 a^3 - 27 b^3.$ |
| 3. $a^3 - 8 b^3.$ | 11. $1 - x^6.$ | 19. $1 - 729 x^3.$ |
| 4. $8 a^3 - 8 b^3.$ | 12. $x^9 - 8.$ | 20. $x^6 - y^{12}.$ |
| 5. $3^3 - 2^3.$ | 13. $1 - 125 x^3.$ | 21. $27 r^3 - 125 s^3.$ |
| 6. $1 - a^{10}.$ | 14. $8 - 27 x^3.$ | 22. $125 x^6 - 81 y^2.$ |
| 7. $1 - 8 a^3.$ | 15. $27 x^3 - 64.$ | 23. $729 - 16 x^3.$ |
| 8. $64 x^3 - y^3.$ | 16. $2^3 x^3 - 1.$ | 24. $125 a^6 - 64 b^9.$ |

25. Also factor Exs. 10, 11, and 20 as the difference of two squares.

26. Find whether $x^3 - y^3$ is exactly divisible by $x + y$, by $x^2 + y^2$, by $x^2 - y^2$. What binomial is a divisor of the difference of two cubes? What is the quotient?

TRINOMIALS OF THE FORM $x^2 + (a+b)x + ab$

153. In § 92 were found such products as

$$(1) (x+5)(x+2) = x^2 + 7x + 10.$$

$$(2) (x-5)(x-2) = x^2 - 7x + 10.$$

$$(3) (x+5)(x-2) = x^2 + 3x - 10.$$

$$(4) (x-5)(x+2) = x^2 - 3x - 10.$$

All these are included in the form

$$(x+a)(x+b) = x^2 + (a+b)x + ab,$$

in which the *coefficient of x is the algebraic sum of a and b and the last term is their product.*

154. It is possible to recognize such products at sight, and thus to find the factors by inspection.

Illustrative Examples. Determine whether the following trinomials are of the kind just considered:

1. $x^2 + 7x + 12$. The question is whether two numbers can be found such that their sum is $+7$ and their product 12. The numbers 3 and 4 answer these conditions. Hence,

$$x^2 + 7x + 12 = (x+3)(x+4).$$

2. $x^2 - 5x - 14$. Since the product of the numbers sought is -14 , one number must have the sign $-$ and the other $+$; and since their sum is -5 , the one having the greater absolute value must have the sign $-$. Hence, the numbers are -7 and $+2$, and we have $x^2 - 5x - 14 = (x-7)(x+2)$.

3. $x^2 - 7x + 12 = (x-3)(x-4)$. Since $(-3)(-4) = +12$ and $(-3) + (-4) = -7$.

4. $x^2 + 4x - 12 = (x+6)(x-2)$. Since $(+6)(-2) = -12$ and $(+6) + (-2) = +4$.

It is to be noted that it is not always possible to find integers to fulfill these two conditions.

The product of two binomials having the same first terms = product of the last terms + ...

TRINOMIALS OF THE FORM $x^2 + (a+b)x + ab$ 155

E.g. Given $x^2 + 5x + 3$. By inspection, it is easily seen that there are no two integers such that their sum is + 5 and their product + 3.

Tell the steps to be taken in deciding whether $x^2 + ax + b$ has two binomial factors.

EXERCISES

Determine whether each of the following trinomials can be factored by inspection, and if so, find the factors.

- | | |
|--------------------------|--------------------------------|
| 1. $x^2 + 3x + 2$. | 26. $a^2 - 14a - 51$. |
| 2. $x^2 + x - 6$. | 27. $a^2 - 3a - 54$. |
| 3. $x^2 - x - 6$. | 28. $x^4 - 8x^2 - 32$. |
| 4. $x^2 - 6x + 8$. | 29. $a^6 - 3a^3 - 154$. |
| 5. $x^2 + 6x + 8$. | 30. $x^2 - 10x + 25$. |
| 6. $x^2 - 3x - 8$. | 31. $a^2b^6 - 13ab^3 - 30$. |
| 7. $x^2 + 2x - 8$. | 32. $x^3 - 17xyz + 72y^2z^2$. |
| 8. $a^2 - 4a - 32$. | 33. $r^3 + 6r^2s - 91s^2$. |
| 9. $a^2 + 4a - 32$. | 34. $a^4c^4 + 9a^2c^2 - 162$. |
| 10. $b^2 + 15b + 56$. | 35. $a^3 + 11a - 210$. |
| 11. $b^2 + 8b + 15$. | 36. $m^4 + 4m^2n^2 + 4n^4$. |
| 12. $b^2 - b - 56$. | 37. $s^2t^2 - 15st - 54$. |
| 13. $b^2 + b - 56$. | 38. $a^2b^2 - 27ab + 26$. |
| 14. $c^2 - 3c - 15$. | 39. $l^2 + 13l + 42$. |
| 15. $x^2 - 15x + 56$. | 40. $x^2y^2 - 11xy - 180$. |
| 16. $x^2 + 15x - 54$. | 41. $9a^2 + 24a + 16$. |
| 17. $x^2 - 14x - 95$. | 42. $81a^2 - 99a + 30$. |
| 18. $y^2 + 21y + 98$. | 43. $g^2 + 26g + 133$. |
| 19. $y^2 - 7y - 98$. | 44. $x^2 + 5xy - 84y^2$. |
| 20. $x^2 - 19x + 78$. | 45. $r^3 + 3r - 154$. |
| 21. $x^4 + 18x^2 + 77$. | 46. $u^2 + 38uv + 165v^2$. |
| 22. $x^4 - 5a^2 - 104$. | 47. $(a+b)^2 - 19(a+b) + 88$. |
| 23. $a^2 + 32a + 240$. | 48. $(x-y)^2 - 14(x-y) + 40$. |
| 24. $a^4 - 11a^2 + 28$. | 49. $(r-s)^2 - 17(r-s) + 60$. |
| 25. $a^4 - 11a^2 - 60$. | 50. $x^2 + (a+b)x + ab$. |

TRINOMIALS OF THE FORM $ax^2 + bx + c$ 155. Find the product of $2x + 5$ and $3x + 2$.

$$\begin{array}{r}
 2x + 5 \\
 3x + 2 \\
 \hline
 6x^2 + 15x \\
 4x + 10 \\
 \hline
 6x^2 + 19x + 10
 \end{array}$$

The products $3x \cdot 2x$ and $2 \cdot 5$ are called *end products* and $2 \cdot 2x$ and $5 \cdot 3x$ are called *cross products*. In this case the cross products are similar with respect to x and are added. Hence the *final product is a trinomial two of whose terms are the end products and the third term is the sum of the cross products*.

This fact enables us to write such products at once.

E.g. $(3a + 4)(5a - 7) = 15a^2 - a - 28$.

In this case $15a^2$ is the first end product and -28 the second, while $-a$ is the sum of the two cross products, $20a$ and $-21a$.

EXERCISES

In this manner obtain the following products:

- | | |
|------------------------------------|--------------------------------------|
| 1. $(2a + 3)(a + 3)$. $+ 1$ | 13. $(3x - 2y)(x + 3y)$. $+ 7xy$ |
| 2. $(4a - 1)(3a + 2)$. $+ 5a$ | 14. $(4a - 3y)(a + y)$. $+ ay$ |
| 3. $(2x + 5)(x - 7)$. $- 9x$ | 15. $(3r - 2s)(2r + s)$. $- 1rs$ |
| 4. $(7r + 8)(3r - 6)$. $- 6$ | 16. $(5m - n)(2m + n)$. $+ 3mn$ |
| 5. $(2x + 8)(9x - 4)$. $+ 16xy$ | 17. $(5a + 3x)(3a - 4x)$. $- 11ax$ |
| 6. $(3m - 1)(4m + 3)$. $+ 5m$ | 18. $(4a - 5b)(a + 3b)$. $+ 7ab$ |
| 7. $(5s - 7)(2s - 4)$. $- 34s$ | 19. $(3a + 5b)(a - b)$. $+ 2ab$ |
| 8. $(2x - 1)(7x + 4)$. $+ x$ | 20. $(3c - 7d)(2c + 3d)$. $- 5cd$ |
| 9. $(4n - 9)(5n - 7)$. $- 77n$ | 21. $(2a - 3b)(3a + 2b)$. $- 5ab$ |
| 10. $(8y - 1)(5y + 11)$. $+ 11y$ | 22. $(6x - 5y)(2x + 3y)$. $+ 12xy$ |
| 11. $(t - 5)(t + 4)$. $- 6$ | 23. $(5a + 3b)(2a - 5b)$. $- 15b^2$ |
| 12. $(5x - y)(2x - 3y)$. $- 11xy$ | 24. $(7m + 5n)(3m + 4n)$. $+ 43mn$ |

156. Trinomials in the form of the above products may sometimes be factored by inspection.

Ex. 1. Factor: $5x^2 + 16x + 3$.

If this is the product of two binomials they must be such that the end products are $5x^2$ and 3 and the sum of the cross products $16x$.

One pair of binomials having the required end products is $5x + 3$ and $x + 1$, others are $5x - 1$ and $x - 3$, $5x + 1$ and $x + 3$, and $5x - 3$ and $x - 1$.

It is convenient to write down these possible pairs of factors as follows, as if arranged for multiplication:

$$\begin{array}{cccc} 5x + 3 & 5x - 1 & 5x + 1 & 5x - 3 \\ x + 1 & x - 3 & x + 3 & x - 1 \end{array}$$

The sum of the cross products in the first pair is $8x$, in the second pair $-16x$, in the third pair $16x$, and in the fourth $-8x$. Since $16x$ is the middle term required, the factors are $5x + 1$ and $x + 3$.

Ex. 2. Factor: $6a^2 - 5a - 4$.

In this case, as in the one preceding, there are several pairs of binomials whose end products are $6a^2$ and -4 , such as $2a - 2$ and $3a + 2$, $6a - 1$ and $a + 4$, etc. By trial we find that among these $3a - 4$ and $2a + 1$ is the only pair the sum of whose cross products is $-5a$. Hence $6a^2 - 5a - 4 = (3a - 4)(2a + 1)$.

In this manner factor the following:

1. $3x^2 + 5x + 2$.
2. $9a^2 + 9a + 2$.
3. $2x^2 + 11x + 12$.
4. $9x^2 + 36x + 32$.
5. $2x^2 - x - 28$.
6. $12s^2 + 11s + 2$.
7. $6t^2 + 7t - 3$.
8. $6x^2 - x - 2$.
9. $5r^2 + 18r - 8$.
10. $14a^2 - 39a + 10$.
11. $5x^2 + 26x - 24$.
12. $2x^2 - 5x + 2$.
13. $2m^2 - m - 3$.
14. $7c^2 - 3c - 4$.
15. $5x^4 + 9x^2 - 18$.
16. $7a^4 + 123a^2 - 54$.
17. $6c^2 - 19c + 15$.
18. $3a^2 - 21a + 30$.
19. $6d^2 + 4d - 2$.
20. $20a^2 - a - 99$.
21. $12c^2 + 25c + 12$.
22. $8 + 6a - 5a^2$.
23. $15 - 5x - 10x^2$.
24. $6 - 5x - 4x^2$.
25. $3h^2 - 13h + 14$.
26. $15r^2 - r - 2$.
27. $2t^2 + 11t + 5$.
28. $10 - 5x - 15x^2$.
29. $5x^2 - 33x + 18$.
30. $20 - 9x - 20x^2$.

FACTORS FOUND BY GROUPING

157. Another method of general application is here applied to polynomials of four terms.

Ex. 1. Find the factors of $ax + ay + bx + by$.

By Principle I, the first two terms may be added and also the last two.

Thus, $ax + ay + bx + by = a(x + y) + b(x + y)$.

These two compound terms have a *common factor*, $(x + y)$, and may be added with respect to this factor by Principle I.

Thus, $a(x + y) + b(x + y) = (a + b)(x + y)$.

Hence, $ax + ay + bx + by = (a + b)(x + y)$.

Ex. 2. Factor $ax - ay - bx + by$.

Combining the first two terms with respect to a and the second two with respect to $-b$, we have,

$$ax - ay - bx + by = a(x - y) - b(x - y).$$

Again combining with respect to the factor $x - y$,

$$ax - ay - bx + by = (a - b)(x - y).$$

The success of this method depends upon the possibility of so grouping and combining the terms as to reveal a *common binomial factor*.

EXERCISES

Factor the following:

- | | |
|----------------------------------|-----------------------------------|
| 1. $ab^2 + ac^2 - db^2 - dc^2$. | 11. $2n^2 - cn + 2nd - cd$. |
| 2. $6ms - 15nt + 9ns - 10mt$. | 12. $5ax - 15ay - 3bx + 9by$. |
| 3. $8ax - 10ay + 4bx - 5by$. | 13. $3xa - 12xc - a + 4c$. |
| 4. $2a^2 + 3ak - 14an - 21nk$. | 14. $3xy - 4mn + 6my - 2xn$. |
| 5. $ac + bc + ad + bd$. | 15. $7mn + 7mr - 2n^2 - 2nr$. |
| 6. $ax^2 - bx^2 - ay^2 + by^2$. | 16. $a - 1 + a^3 - a^2$. |
| 7. $8ac - 20ad - 6bc + 15bd$. | 17. $3s + 2 + 6s^4 + 4s^2$. |
| 8. $2ax - 6bx + 3by - ay$. | 18. $as^2 - 3bst - ast + 3bt^2$. |
| 9. $5 + 4a - 15c - 12ac$. | 19. $3mn + 6m^2 - 2am - an$. |
| 10. $15b - 6 - 20bc + 8c$. | 20. $2ar + 2as + 2br + 2bs$. |

SQUARES OF POLYNOMIALS

158. **Example.** By multiplication find the square of $a + b + c$, and reduce the result to simplest form.

How many terms are there in the product? How many are squares? How many are of the type $2ab$?

From this we get the following rule:

The square of a ^{polynomial}trinomial consists of the sum of the squares of its terms plus twice the product of each term by each succeeding term.

In symbols this is

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

EXERCISES

1. How may the above rule be applied to find the square of $a - b + c$, of $a + b - c$, of $a - b + c$?

Suggestion. $a - b + c = a + (-b) + c$.

2. Find the square of $(2a + b - 3c)$.

Suggestion. $(2a + b - 3c)^2 = (2a)^2 + b^2 + (-3c)^2 + 2(2a)b + 2(2a)(-3c) + 2 \cdot b(-3c)$. This should now be simplified.

Find the squares of the following:

3. $a + 2b + 3c$. 6. $2x + 3y - z$. 9. $5a + 2b - 3c$.

4. $2a + b + 4c$. 7. $4x - y + 2$. 10. $a - b - 2c$.

5. $x - 3y - z$. 8. $3a - b + c$. 11. $x - 3y - z$.

12. By multiplication find the square of $a + b + c + d$, and reduce to simplest form. Study this product and try to make a rule for squaring any polynomial.

159. In the preceding exercises, the square of each trinomial consists of six terms, namely *three squared terms and three sums of cross products*.

It follows that the square root of such a polynomial may be found by taking the square roots of the three squared terms and *determining their signs by trial* in such a way as to give the proper cross products.

Ex. Find the square root of

$$4x^2 - 12xy - 16xz + 9y^2 + 24yz + 16z^2.$$

Solution. The terms $4x^2$, $9y^2$, and $16z^2$ are all squares. The square root of $4x^2$ is either $+2x$ or $-2x$, that of $9y^2$ is either $+3y$ or $-3y$, that of $16z^2$ is $+4z$ or $-4z$. Hence if the given polynomial is a perfect square, the terms $-12xy$, $-16xz$, and $24yz$ must be the sums of the cross products of $2x$, $3y$, and $4z$ each taken with the proper sign. By inspection we soon find that the sign of $2x$ must be $+$, and that of $3y$ and $4z$ each $-$.

Hence the required square root is $2x - 3y - 4z$.

Is every polynomial of six terms the square of a trinomial? In order to be such a square, how many squared terms must there be? What is the sign of each squared term and why? How are the signs of the square roots of these squared terms determined?

EXERCISES

In each of the following determine whether the polynomial is a perfect square and if so find its square root:

1. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz$.
2. $a^2 - 8ab + 16b^2 - 2ac + c^2 + 8bc$.
3. $9x^2 + 4y^2 + z^2 - 12xy + 6xz - 4yz$.
4. $y^4 - 4y^2 - 8xy^2 + 16x + 16x^2 + 4$.
5. $a^3 + a^2b^2 - 2a^2b + 2abc - ab^2c + b^2c^2$.
6. $x^6 - 4x^5 + 4x^4 + 6x^3 - 12x^2 + 9$.
7. $x^3 + 16x^2y^2 + 289 + 8x^2y + 34x + 136xy$.
8. $9a^4 + 4a^2 + 256 - 12a^3 - 96a^2 + 64a$.
9. $16x^6 + 4y^2 + 1 - 16x^2y + 8x^3 - 4y$.
10. $25 + 49x^2 + 4x^4 - 70x - 20x^3 + 28x^5$.
11. $a^4b^4 - 2a^2b^2 + a^2 - 2a^2b^2 + 2ab + b^2$.

Besides the methods of factoring which have been applied to the types of expressions thus far considered, there are various other processes which will be considered in the Advanced Course.

SUMMARY OF FACTORING

1. Tell how to decide whether a polynomial has a monomial factor. Give an example of such a polynomial and find its factors.

2. Tell how to decide whether a binomial is the difference of two squares. Give such a binomial and find its factors.

3. Tell how to decide whether a trinomial is a perfect square. Distinguish two kinds. Give a trinomial square of each kind and factor it.

4. Give a rule for factoring the sum of two cubes.

5. Give a rule for factoring the difference of two cubes.

6. Tell how to decide whether a trinomial of the form $x^2 + bx + c$ is the product of two binomials.

7. Tell how to decide whether a trinomial of the form $ax^2 + bx + c$ is the product of two binomials.

8. Tell how to factor a polynomial by grouping. Give a polynomial which can be factored in this way and find its factors.

9. Describe the square of a trinomial. Tell how to decide whether a polynomial is a perfect square.

MISCELLANEOUS EXERCISES

Classify the following expressions according to the foregoing types for factoring, and find the factors:

1. $x^2 + 5x + 6$.

7. $2n^2 - 6nc - 3ny + 9cy$.

2. $1 - x^3$.

8. $4x^2 - y^2$.

3. $x^2 + 11x + 30$.

9. $a^3 + b^3$.

4. $4x^2 + 9y^2 + 12xy$.

10. $9x^2 + y^4 + 6xy^2$.

5. $4x^2 + 9y^2 - 12xy$.

11. $2y^2a^3 + 4ya^2 - 8ya$.

6. $5x^2 + 4ax + 7xy$.

12. $x^2 + 7x + 6$.

1. 13. $9x^2 + 36y^4 + 36xy^2$.
2. 14. $9y - 9z - 2xy + 2xz$.
3. 15. $a^2 - 1$.
4. 16. $a^4 + b^4 + 2a^2b^2$.
5. 17. $a^4 - 25$.
6. 18. $27a^3 - 125$.
7. 19. $4a^2 + 4ab + ab^2$.
8. 20. $4a^2 + 9x^4 - 12ax^2$.
9. 21. $1 + x^2$.
10. 22. $2x^2 + 5x + 3$.
11. 23. $36 + 4x^6 + 24x^3$.
12. 24. $(x-1)^2 - (x+1)^2$.
13. 25. $8 + 64a^2$.
14. 26. $ac - ax - 4bc + 4bx$.
15. 27. $27 - 216a^3$.
16. 28. $3^3 + 6^3a^3$.
17. 29. $25(x+1)^2 - 4$.
18. 30. $5cx - 10c + 4dx - 8d$.
19. 31. $4(x+2)^2 + y^2 + 4(x+2)y$.
20. 32. $ra + 2rh - 5sa - 10sh$.
21. 33. $-2a^2b + a^4 + b^2$.
22. 34. $2ha - hb + 6a - 3b$.
23. 35. $3(a+1)^3 + 4(a+1)^2 + a + 1$.
24. 36. $(x+a)^2 - (x-a)^2$.
25. 37. $15m^2 + 224m - 15$.
26. 38. $3x^2 + 27x + 42$.
27. 39. $x^4 + 49a^2 + 14ax^2$.
28. 40. $27a^6 - a^3x^3$.
29. 41. $3^3a^6 + a^3x^3$.
30. 42. $8bd - 40be + 3cd - 15ce$.
31. 43. $x^2 - 11x + 30$.
32. 44. $(x+2)^2 - 4(x-2)^2$.
33. 45. $x^4 + 9y^2 - 6yx^2$.
34. 46. $4a^2 - 7ca^2 - 4d^2 + 7cd^2$.
35. 47. $a^2 + 15a - 16$.
36. 48. $18 - 27c + 16b - 24cb$.
37. 49. $4 - (a^2 + b^2 - 2ab)^2$.
38. 50. $10r + 3bs - 6br - 5s$.
39. 51. $25 + 64x^6 + 80x^3$.
40. 52. $1000 - x^3$.
41. 53. $10^3 + x^3$.
42. 54. $8a^3 + a^3b^3 + b^2a^2$.
43. 55. $100 - 49x^4$.
44. 56. $100 + 625 + 500$.
45. 57. $a^2 - 17a + 72$.
46. 58. $a^2 + 17a + 72$.
47. 59. $a^2 + 16b^2 - 8ab$.
48. 60. $x^5 - y^3$.
49. 61. $24a^3 + 37a - 72$.
50. 62. $x^4 + 15x^2 - 100$.
51. 63. $9^6 + 8^3$.
52. 64. $9x^4 + 16y^2 + 24x^2y$.
53. 65. $1 - 1000$.
54. 66. $16a^2b^2 + 24ab + 36b^3$.
55. 67. $64 + 8$.
56. 68. $16a^2b^2 + 9a^2c^2 + 24a^2bc$.

69. $a^2 + 4b^2 + 4ab - 4x^2$.
 70. $a^3b^6 + c^3$.
 71. $5x^2 + 10x^2y^2 + 30x^2y^4$.
 72. $16a^2c^2 + 4c^2x^2 + 16ac^2x$.
 73. $a^6y^3 - z^3$.
 74. $x^4 - 7x^2 - 120$.
 75. $9a^4b^2 - 12a^2b + 4a^2$.
 76. $8ab + 27ab^7$.
 77. $x^4 + 4x^2 + 4 - x^6$.
 78. $1 - 125a^3b^6$.
 79. $16 + 16ab + 4a^2b^2$.
 80. $64a^3 + 8a^2b^3$.
 81. $65r^2 + 8r - 1$.
 82. $a^2 - 13a - 140$.
 83. $x^6 + 17x^3 + 30$.
 84. $25 - (a^4 - 2a^2b^3 + b^6)$.
 85. $36a^2 - 29ab + 5b^2$.
 86. $a^2 - a - 380$.
 87. $24a^4c^4 + a^6 + 144c^3a^2$.
 88. $9x^2 + 4y^4 - 12xy^2 - 16$.
 89. $81 + 100x^3 - 180x^4$.
 90. $a^4 + 27a^2 + 180$.
 91. $a^4 + 3a^2 - 180$.
 92. $a^4 - 3a^2 - 180$.
 93. $144 - (a^4 + b^2 - 2a^2b)$.
 94. $81a^2b^4 + 49c^3 - 126ab^2c$.
 95. $12s^2 - 23st + 10t^2$.
 96. $36x^4 + 12x^2y^4 + y^8$.
 97. $16x^6 + 9y^4 + 24x^2y^2 - 49$.
 98. $y^2 + 35y + 300$.
 99. $5y^2 - 80y + 300$.
 100. $39x^4 - 16x^2 + 1$.
 101. $ac - bc + ad - bd$.
 102. $625 - (31 - 4a^2)^2$.
 103. $z^3 + ya - y^2z^3 - ay^4$.
 104. $x^4 + 2x^2 + 1 - x^2$.
 105. $60x^2 + 7xy - y^2$.
 106. $x^2 - 20xy + 75y^2$.
 107. $x^2 - 17x - 60$.
 108. $36a^2b^4 + c^2b^4 + 12ab^4c$.
 109. $4a^2 + 9b^4 + 12ab^2 - 16a^4$.
 110. $100 - (16x^2 + y^6 - 8xy^3)$.
 111. $6rd - 15re + 22cd - 55ce$.
 112. $-112a^2c^3 + 49a^4 + 64c^6$.
 113. $4x^2 + 9y^2 + z^2 - 12xy - 4xz + 6yz$.
 114. $a^2b^2 + a^2c^2 + b^2c^2 - 2a^2bc + 2acb^2 - 2abc^2$.
 115. $4a^2 - 12ab + 4ac + 9b^2 + c^2 - 6bc$.
 116. $a^6 + 10a^3 + 9a^2 + 25 - 6a^4 - 30a$.

REVIEW QUESTIONS

1. Define a positive integral exponent. Explain the difference between an exponent and a coefficient.

2. Show how the multiplication of monomials may depend upon Principle XIII.

3. Under what circumstances are exponents added in multiplication? State Principle XVI. Use this principle to show that $(a^2)^3 = a^6$, $(a^3)^4 = a^{12}$.

4. Under what circumstances are exponents subtracted in division? State Principle XVII.

5. Show how the division of monomials may depend upon Principles XV and XVII.

6. What is meant by factoring? Is the following expression factored? $x(a+b) + y(a+b)$. Why?

7. What are the characteristics of a trinomial square?

Are the following trinomials squares? If not, state what is lacking. $x + xy + y^2$; $x^4 + x^2y^2 + y^4$; $a^2 - 2ab - b^2$; $4a^2 + 4ab + 4b^2$.

8. What are the factors of the difference of two squares? Factor $x^6 - y^6$ as the difference of two squares.

9. What are the factors of the sum of two cubes? Factor $x^6 + y^6$ as the sum of two cubes.

10. What are factors of the difference of two cubes? Factor $x^6 - y^6$ as the difference of two cubes.

11. Explain how to factor a trinomial by inspecting the end products and cross products of two binomials.

12. By means of the following examples explain the process of factoring by grouping.

$$x^2 + ax + bx + ab; \quad x^3 - x - 3x^2 + 3.$$

13. What are the characteristics of the square of a trinomial? Of the square of *any* polynomial?

14. State Principles XVI and XVII as formulas and add them to your list.

DRILL EXERCISES

1. Find the average of the following temperatures: 7 A.M., -4° ; 8 A.M., -2° ; 9 A.M., -1° ; 10 A.M., $+1^{\circ}$; 11 A.M., $+5^{\circ}$; 12 M., $+7^{\circ}$.

2. $7x + (8x + 4) + 2 = 4x + 9$.

✓ 3. $6x + 4(4x + 2) = 85 - 3(2x + 7)$.

✓ 4. $8 + 7(6 + 6n) + 2n = 2(4n + 5) + 18n + 49$.

5. $5(9x + 3) + 6x = 24x - 4(3x + 2) + 36$.

✓ 6. $\frac{7(12x + 8)}{4} + 13 + 5x - 6 = 47$.

7. $\frac{a-b}{2x-1} = \frac{4c}{(2x+1)(2x-1)} - \frac{a-b}{2x+1}$.

8. $\frac{x+2}{a} = \frac{x-1}{b} + \frac{3x+2}{ab}$.

✓ 9. $\begin{cases} 6y - x = 7 + 4y, \\ 5x + 8y = 1. \end{cases}$

12. $\begin{cases} 5x + 3y = 0, \\ 2x + y = 1. \end{cases}$

10. $\begin{cases} x - y = 37, \\ 2x + 3y = 314 + 13y. \end{cases}$

13. $\begin{cases} 2x + 3y = 6x - 1, \\ 3x - 2y = 3. \end{cases}$

✓ 11. $\begin{cases} 2x - 3y = y + 6, \\ x - 2y = 4y + 3. \end{cases}$

✓ 14. $\begin{cases} 5x - 3y = 0, \\ 2x + 2 - 6y = 2 - x. \end{cases}$

15. $\begin{cases} x + 2y + 2z = 3, \\ 3x - 4y + z = 19, \\ -2x + 6y + 3z = 0. \end{cases}$

16. $\begin{cases} 2x + 5y + 7z = 7, \\ 3x - 9y - 2z = 23, \\ -x + 3y + 3z = -10. \end{cases}$

✓ 17. Graph the equations: $x - 4y = 7$ and $2x + y = 4$.

18. The older of two sisters is now 8 years less than twice as old as the other. If x represents the age of the younger sister, represent in symbols twice the sum of their ages 7 years ago.

19. A rear wheel of a wagon has a circumference 4 feet greater than that of a front wheel. If x represents the number of feet in the circumference of the rear wheel, represent in symbols the number of revolutions each wheel must make to go one mile.

CHAPTER X

EQUATIONS SOLVED BY FACTORING

160. Illustrative Problem. There are two consecutive integers the sum of whose squares is 61. What are the numbers?

Solution. If x = one of the numbers, then $x + 1$ is the other.

Hence,
$$x^2 + (x + 1)^2 = 61 \quad (1)$$

By F ,
$$x^2 + x^2 + 2x + 1 = 61 \quad (2)$$

By F, I, S ,
$$2x^2 + 2x = 60 \quad (3)$$

By D ,
$$x^2 + x = 30 \quad (4)$$

Equation (4) differs from any which we have studied heretofore in that it contains the square of the unknown number, after all possible reductions have been made.

161. Definition. Equations which involve the second but no higher degree of the unknown number are called **quadratic equations**.

One method of solving quadratic equations is now to be considered.

By S , equation (4) above may be written

$$x^2 + x - 30 = 0. \quad (5)$$

Factoring the left member,

$$(x + 6)(x - 5) = 0. \quad (6)$$

This equation is satisfied by $x = 5$ since $(5 + 6)(5 - 5) = 11 \cdot 0 = 0$, and also by $x = -6$ since $(-6 + 6)(-6 - 5) = 0 \cdot (-11) = 0$.

Test by substitution whether 5 and $5 + 1 = 6$ satisfy the conditions of the problem; also whether -6 and $-6 + 1 = -5$ satisfy it.

It thus appears that equation (4) has two roots, namely, 5 and -6 . The two pairs of corresponding integers 5, 6 and -6 , -5 each satisfy the conditions of the problem.

EXERCISES

1. If one of two factors is zero, what is the product? Does it matter what the other factor is?

2. Find a value of x which makes $(x-3)(x+2)$ equal to zero. Does this value of x make both factors equal to zero? Is it necessary that both factors should be made equal to zero?

3. Find a value of x which satisfies the equation

$$(x-7)(x^2+2x-3)=0;$$

also one which satisfies $(x+8)(x^3+x+4)=0$.

Suggestion. Find a value of x which makes the first factor zero in each case.

4. Find two values of x which satisfy $(x-3)(x+4)=0$, also two which satisfy $(x+8)(x-3)=0$.

5. Find two values of x which satisfy $5x(x+7)=0$. Does $x=0$ satisfy this equation?

6. Find two values of x which satisfy $(3x-2)(2x+5)=0$.

162. The method of solution suggested by the foregoing examples consists of three steps:

(1) *Transform the equation so that all terms are collected in one member, with similar terms united, leaving the other member zero.* This can always be done by Principle VI. It is convenient to make the *right* member zero.

(2) *Factor the expression on the left.*

(3) *Find the values of x which make each of these factors zero.* This is readily done by setting each factor equal to zero and solving it for the unknown.

EXERCISES

Find two solutions for each of the following quadratic equations:

1. $x^2 - 3x + 2 = 0$.

5. $a^2 + 10a + 8 = -3a - 34$.

2. $x^2 + 7x = 30$.

6. $a^2 + 3a = 10a + 18$.

3. $a^2 - 11a = -30$.

7. $a^2 + 10a = -24 - 4a$.

4. $a^2 + 13a = 30$.

8. $2x^2 - 6x = -40 + 12x$.

9. $3x + x^2 = 20x - 72$. 12. $11x + 3x^2 = 20$.
 10. $17x + 30 = -x^2 - 40$. 13. $16 - 5x + x^2 = -2x^2 -$
 11. $7x^2 + 2x = 30x - 21$. $20x - 2$.
 14. $x^2 - 16 = 0$. 15. $x^2 - 1 = 0$. 16. $x^2 - x = 0$. 17. $x^2 + x = 0$.
 18. $4x^2 = 25$. 19. $x^2 + 4x + 4 = 0$. 20. $x^2 + 8x + 16 = 0$.
 21. $x^2 + 12x + 6 = 5x - 4$. 23. $60x + 4x^2 + 144 = 8x$.
 22. $2x^2 - 7x = 60 + 7x$. 24. $18x = 63 - x^2$.
 25. $24x^2 = 12x + 12$. 27. $22x + x^2 = 363$. 29. $2x^2 = 2 - 3x$.
 26. $2x = 63 - x^2$. 28. $3x^2 + 7x = 6$. 30. $x - 2 = -3x^2$.

163. It is sometimes possible by the above process to solve equations in which the exponent of the highest power of the unknown is greater than 2.

Ex. 1. Solve the equation:

$$x^3 + 30x = 11x^2. \quad (1)$$

By S, $x^3 - 11x^2 + 30x = 0. \quad (2)$

By § 141, $x(x^2 - 11x + 30) = 0. \quad (3)$

By § 154, $x(x - 5)(x - 6) = 0. \quad (4)$

(4) is satisfied if $x = 0$, if $x - 5 = 0$, and if $x - 6 = 0$.

Hence the roots are $x = 0$, $x = 5$, $x = 6$.

Ex. 2. $x(x+1)(x-2)(x+3) = 0$.

Any value of x which makes one of these factors zero reduces the product to zero and hence satisfies the equation. Hence the roots of the equation are found from

$$x = 0, x + 1 = 0, x - 2 = 0, \text{ and } x + 3 = 0.$$

That is, $x = 0$, $x = -1$, $x = 2$, $x = -3$ are the values of x which satisfy the equation.

Notice that this process is applicable only when one member of the equation is *zero* and the other member is *factored*.

Solve the following equations by factoring:

1. $x^3 - x^2 = 6x$. 3. $x^3 - 25x = 0$.

2. $5x = 4x^2 + x^3$. 4. $x^3 - 3x^2 = -2x$.

5. $3x^3 = 15x^2 + 42x$.
 6. $5x^3 + 315x = 80x^2$.
 7. $x^2 + ax + bx + ab = 0$.
 8. $x^2 - ax - bx + ab = 0$.
 9. $4(x-2)^2 - (x+3)^2 = 0$.
10. $x^2 - ax + bx - ab = 0$.
 11. $x^2 + ax - bx - ab = 0$.
 12. $9(x+2)^2 - 4(x-3)^2 = 0$.
 13. $x^3 - x - 3x^2 + 3 = 0$.
 14. $x^3 - 4x - 8x^2 + 32 = 0$.

PROBLEMS SOLVED BY FACTORING

164. Illustrative Problem. The paving of a square court costs 40 ¢ per square yard and the fence around it costs \$ 1.50 per linear yard. If the total cost of the pavement and the fence is \$ 100, what is the size of the court?

Solution. Let x = the length of one side in yards.
 Then $40x^2$ = cost in cents of paving the court,
 and $150 \cdot 4x = 600x$ = cost of the fence in cents.

$$\text{Hence,} \quad 40x^2 + 600x = 10000. \quad (1)$$

$$\text{By } D, \quad x^2 + 15x = 250. \quad (2)$$

$$\text{By } S, \quad x^2 + 15x - 250 = 0. \quad (3)$$

$$\text{Factoring,} \quad (x-10)(x+25) = 0. \quad (4)$$

$$\text{Whence,} \quad x = 10, \text{ and also } x = -25. \quad (5)$$

It is clear that the length of a side of the court cannot be -25 yards. Hence 10 is the only one of these two results which has a meaning in this problem.

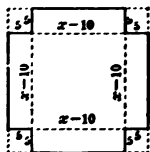
- It happens frequently when a quadratic equation is used to solve a problem that one of the two numbers which satisfy this equation will not satisfy the conditions of the problem.

PROBLEMS

In each of the following problems find all the solutions possible for the equations and then determine whether or not each solution has a reasonable interpretation in the problem.

1. The dimensions of a picture inside the frame are 12 by 16 inches. What is the width of the frame if its area is 288 square inches?

2. An open box is made from a square piece of tin by cutting out a 5-inch square from each corner and turning up the sides.



How large is the original square if the box contains 180 cubic inches?

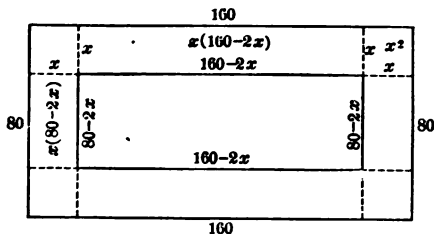
If x = length of a side of the tin, then the volume of the box is: $5(x-10)(x-10)=180$. (See the figure.)

3. A rectangular piece of tin is 4 inches longer than it is wide. An open box containing 840 cubic inches is made by cutting a 6-inch square from each corner and turning up the ends and sides. What are the dimensions of the box?

4. A farmer has a rectangular wheat field 160 rods long by 80 rods wide. In cutting the grain, he cuts a strip of equal width around the field.

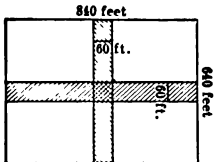
How many acres has he cut when the width of the strip is 8 rods?

5. How wide is the strip around the field of problem 4, if it contains $27\frac{1}{2}$ acres?



6. In the northwest a farmer using a steam plow starts plowing around a rectangular field 640 by 320 rods. If the strip plowed the first day lacks 16 square rods of being 24 acres, how wide is it?

7. A rectangular piece of ground 840 by 640 feet is divided into 4 city blocks by two streets 60 feet wide running through it at right angles. How many square feet are contained in the streets?



8. A farmer lays out two roads through the middle of his farm, one running lengthwise of the farm and the other crosswise. How wide are the roads if the farm is 320 by 240 rods, and the area of the roads is 1671 square rods?

QUADRATIC AND LINEAR EQUATIONS

165. When two simultaneous equations are given, one quadratic and one linear, they may be solved by the process of substitution, which was used (§ 122) in the case of two linear equations.

Illustrative Example. Solve the equations:

$$\begin{cases} x^2 - y^2 = -16. & (1) \end{cases}$$

$$\begin{cases} x - 3y = -12. & (2) \end{cases}$$

From (2) by S , $x = 3y - 12.$ (3)

Substituting (3) in (1), $(3y - 12)^2 - y^2 = -16.$ (4)

From (4) by F , $9y^2 - 72y + 144 - y^2 = -16.$ (5)

From (5) by F , A , $8y^2 - 72y + 160 = 0.$ (6)

By D , $y^2 - 9y + 20 = 0.$ (7)

Factoring, $(y - 5)(y - 4) = 0.$ (8)

Hence, $y = 5$, and $y = 4.$ (9)

Substitute $y = 5$ in (3) and find $x = 3.$

Substitute $y = 4$ in (3) and find $x = 0.$

Therefore (1) and (3) are satisfied by the two pairs of values,

$$x = 3, y = 5, \text{ and } x = 0, y = 4.$$

Check by substituting these pairs of values in (1) and (2).

EXERCISES

In the manner just illustrated solve the following:

1. $\begin{cases} x + 2y = 8, \\ 5x^2 + 12y^2 = 128. \end{cases}$

5. $\begin{cases} x - 2y = -2, \\ x^2 - 6y^2 = 10. \end{cases}$

2. $\begin{cases} x + y = 1, \\ x^2 + y^2 = 1. \end{cases}$

6. $\begin{cases} 8x - 16y = -120, \\ 7x^2 + 2y^2 = 585. \end{cases}$

3. $\begin{cases} 2x - y = 6, \\ 4x^2 + 5y^2 = 36. \end{cases}$

7. $\begin{cases} 7x + 9y = 88, \\ 7x^2 + 9y^2 = 736. \end{cases}$

4. $\begin{cases} x + 3y = 6, \\ x^2 + 3y^2 = 12. \end{cases}$

8. $\begin{cases} x - y = 6, \\ x^2 - 7y^2 = 36. \end{cases}$

- | | |
|------------------------------------------------------------------------|-----------------------------------------------------------------------|
| 9. $\begin{cases} 3x + 2y = 7, \\ 3x^2 + 8y^2 = 35. \end{cases}$ | 15. $\begin{cases} 6x - 7y = 18, \\ 36x^2 - 7y^2 = 324. \end{cases}$ |
| 10. $\begin{cases} x - 8y = -11, \\ 3x^2 - 16y^2 = 11. \end{cases}$ | 16. $\begin{cases} x - 9y = 2, \\ x^2 - 45y^2 = 4. \end{cases}$ |
| 11. $\begin{cases} x - y = -7, \\ 4x^2 + 3y^2 = 147. \end{cases}$ | 17. $\begin{cases} x + y = 8, \\ 13x^2 + 3y^2 = 160. \end{cases}$ |
| 12. $\begin{cases} x - y = 2, \\ x^2 - 5y^2 = 4. \end{cases}$ | 18. $\begin{cases} 2x - 5y = -16, \\ 4x^2 + 15y^2 = 256. \end{cases}$ |
| 13. $\begin{cases} x - y = 1, \\ 3x^2 - 2y^2 = -5. \end{cases}$ | 19. $\begin{cases} 7x + 4y = 7, \\ 49x^2 - 8y^2 = 49. \end{cases}$ |
| 14. $\begin{cases} 5x - 7y = -28, \\ 15x^2 + 49y^2 = 784. \end{cases}$ | 20. $\begin{cases} x - 3y = -12, \\ x^2 - y^2 = -16. \end{cases}$ |

166. If squares are constructed on the two sides, and also on the hypotenuse of a right-angled triangle, then the *sum of the squares on the sides is equal to the square on the hypotenuse*. This is proved in geometry, but may be verified by counting squares in the accompanying figure. This proposition was first discovered by the great philosopher and mathematician Pythagoras, who lived about 550 B.C. Hence it is called the Pythagorean proposition.

We now proceed to solve some problems by this proposition.

PROBLEMS

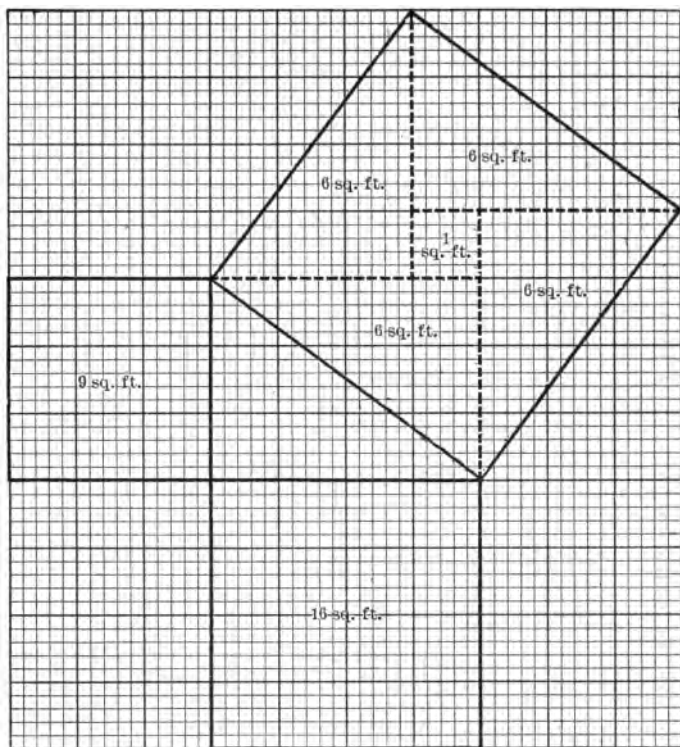
1. The sum of the sides about the right angle of a right triangle is 35 inches, and the hypotenuse is 25 inches. Find the sides of the triangle.

Solution. Let a = the length of one side in inches,
and b = the length of the other.

Then $a + b = 35,$ (1)

and $a^2 + b^2 = 25^2 = 625$ (Pythagorean proposition). (2)

From (1), $a = 35 - b.$



Substituting in (2),

or

$$(35 - b)^2 + b^2 = 625,$$

$$1225 - 70b + b^2 + b^2 = 625,$$

$$2b^2 - 70b + 600 = 0,$$

$$b^2 - 35b + 300 = 0,$$

Factoring,

$$(b - 20)(b - 15) = 0.$$

Whence

$$b = 20, \text{ and } b = 15.$$

From (1), if $b = 20$, $a = 15$, and if $b = 15$, $a = 20$; that is, the sides of the triangle are 15 and 20.

2. The difference between the two sides of a right triangle is 2 feet, and the length of the hypotenuse is 10 feet. Find the two sides.

3. The sum of the length and width of a rectangle is 17 rods, and the diagonal is 13 rods. Find the dimensions of the rectangle.

4. A room is 3 feet longer than it is wide, and the length of the diagonal is 15 feet. Find the dimensions of the room.

5. The length of the molding around a rectangular room is 46 feet, and the diagonal of the room is 17 feet. Find its dimensions.

6. The longest rod that can be placed flat on the bottom of a certain trunk is 45 inches. The trunk is 9 inches longer than it is wide. What are its dimensions?

7. The floor space of a rectangular room is 180 square feet, and the length of the molding around the room is 56 feet. What are the dimensions of the room?

8. A rectangular field is 20 rods longer than it is wide, and its area is 2400 square rods. What are its dimensions?

9. A ceiling requires 24 square yards of paper, and the border is 20 yards long. What are the dimensions of the ceiling?

10. The area of a triangle is 18 square inches, and the sum of the base and altitude is 12. Find the base and altitude.

11. The altitude of a triangle is 7 inches less than the base, and the area is 130 inches. Find the base and altitude.

12. The sum of two numbers is 17, and the sum of their squares is 145. Find the numbers.

13. The difference of two numbers is 8, and the sum of their squares is 274. Find the numbers.

14. The difference of two numbers is 13, and the difference of their squares is 481. Find the numbers.

15. The sum of two numbers is 40, and the difference of their squares is 320. Find the numbers.

16. The sum of two numbers is 45, and their product is 450. Find the numbers.

17. The difference of two numbers is 32, and their product is 833. What are the numbers?

DRILL EXERCISES

1. $\frac{12(5+4x)}{6} - \frac{5(6+4x)}{2} + 50 = x + 18.$
2. $15 + \frac{21(3+x)}{7} + \frac{2(6+18x)}{3} = \frac{3(9x+12)}{3} + 28.$
3. $\frac{11(5x+25)}{5} + \frac{3(6x-2)}{2} = \frac{7(4x+8)}{4} + \frac{12x+36}{3} + 35.$
4. $\frac{2(x+1)}{x-1} - 3 = -\frac{x-1}{x+1}.$
5. $\frac{x+a}{x-a} = 5 - \frac{4(x-a)}{x+a}.$
6. $\begin{cases} \frac{5x-3}{4} - \frac{2y-5}{2} = \frac{1}{2}, \\ \frac{3x+5}{2} + \frac{y-10}{5} = 6. \end{cases}$
7. $\begin{cases} \frac{2x-9}{3} - \frac{4y-2}{5} = -1, \\ \frac{7x+4}{2} + \frac{y+6}{4} = 3. \end{cases}$
8. $\begin{cases} \frac{a+7}{3} + \frac{4b-1}{6} = \frac{1}{2}, \\ \frac{4a+7}{3} + \frac{7b+3}{5} = -5. \end{cases}$
9. $\begin{cases} \frac{2a+5}{3} + \frac{3b-10}{4} = 0, \\ \frac{3a+6}{8} - \frac{5-4b}{7} = 2. \end{cases}$
10. $\begin{cases} \frac{x}{3} + \frac{y}{4} + \frac{z}{6} = 6, \\ \frac{x}{2} + \frac{y}{8} - \frac{z}{12} = 3, \\ \frac{x}{6} - \frac{y}{2} + \frac{z}{3} = 1. \end{cases}$
11. $\begin{cases} x+y+z = 35, \\ \frac{x}{3} + \frac{y}{5} + z = 15, \\ \frac{x}{5} + \frac{y}{5} - z = 3. \end{cases}$

12. If r represents the rate in miles per hour at which a train is moving, how far will it go in t hours? Another train runs 10 miles per hour faster. Express in symbols the sum of the distances which these two trains travel in t hours.

13. If r_1 represents the rate at which a river is flowing and r_2 the rate at which a steamer can go in still water, express in symbols the distance which the steamer can go in t hours: (a) down the river; (b) up the river.

CHAPTER XI

SQUARE ROOTS AND RADICALS

167. Definition. A square root of a number is one of its two equal factors.

Thus 3 is a square root of 9, since $3 \cdot 3 = 9$. Similarly $a + b$ is a square root of $a^2 + 2ab + b^2$.

It should be noted that every square has *two* square roots.

E.g. -3 is also a square root of 9 as well as $+3$, since $(-3) \cdot (-3) = 9$.

The positive square root of a number is indicated by the radical sign $\sqrt{\quad}$ alone or preceded by the sign $+$. The negative square root is indicated by the radical sign preceded by the sign $-$.

E.g. $+\sqrt{9}$ or $\sqrt{9} = +3$ and not -3 , and $-\sqrt{9} = -3$, and not $+3$.

The square root of any number is at once evident if we can resolve it into two equal groups of factors.

E.g.

$$\sqrt{576} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3} = \sqrt{(2^3 \cdot 2)(2^3 \cdot 3)} = \sqrt{24 \cdot 24} = 24.$$

EXERCISES

Find by inspection the following indicated square roots:

- | | | | |
|-------------------|---------------------|----------------------|------------------------|
| 1. $\sqrt{4}$. | 8. $\sqrt{121}$. | 15. $\sqrt{324}$. | 22. $-\sqrt{2^4}$. |
| 2. $\sqrt{9}$. | 9. $-\sqrt{169}$. | 16. $\sqrt{289}$. | 23. $-\sqrt{5^{13}}$. |
| 3. $-\sqrt{16}$. | 10. $\sqrt{225}$. | 17. $-\sqrt{625}$. | 24. $-\sqrt{7^8}$. |
| 4. $\sqrt{25}$. | 11. $\sqrt{196}$. | 18. $-\sqrt{900}$. | 25. $\sqrt{a^{13}}$. |
| 5. $\sqrt{36}$. | 12. $-\sqrt{256}$. | 19. $\sqrt{10000}$. | 26. $-\sqrt{3^{14}}$. |
| 6. $-\sqrt{49}$. | 13. $-\sqrt{576}$. | 20. $\sqrt{a^4}$. | 27. $\sqrt{a^{24}}$. |
| 7. $\sqrt{81}$. | 14. $\sqrt{400}$. | 21. $-\sqrt{x^8}$. | 28. $\sqrt{3^4}$. |

168. The square root of the product of several factors, each of which is a square, may be found in two ways if the factors are expressed in Arabic figures.

$$E.g. \quad \sqrt{4 \cdot 16 \cdot 25} = \sqrt{1600} = \sqrt{40 \cdot 40} = 40,$$

$$\text{or} \quad \sqrt{4 \cdot 16 \cdot 25} = \sqrt{2^2 \cdot 4^2 \cdot 5^2} = 2 \cdot 4 \cdot 5 = 40.$$

But with literal factors, the second process only is available.

$$E.g. \quad \sqrt{16a^2b^4c^2} = \sqrt{4^2a^2(b^2)^2c^2} = 4ab^2c.$$

EXERCISES

Find the following indicated square roots:

- | | | |
|--------------------------------------|----------------------------------|------------------------------|
| 1. $-\sqrt{2^3 \cdot 3^3}$ | 7. $-\sqrt{3^{12} \cdot 5^{14}}$ | 13. $\sqrt{9x^4y^{12}}$ |
| 2. $\sqrt{81 \cdot 121}$ | 8. $-\sqrt{2^{22} \cdot 3^{12}}$ | 14. $-\sqrt{121a^2x^4}$ |
| 3. $\sqrt{49 \cdot 25 \cdot 169}$ | 9. $\sqrt{16a^2b^2c^2}$ | 15. $-\sqrt{7^4a^4b^2}$ |
| 4. $-\sqrt{8^2 \cdot 5^2 \cdot 3^2}$ | 10. $\sqrt{64a^4x^4}$ | 16. $\sqrt{625x^4y^2}$ |
| 5. $\sqrt{5^4 \cdot 3^2 \cdot 4^4}$ | 11. $-\sqrt{4^4a^2b^4}$ | 17. $\sqrt{1225a^{2r}}$ |
| 6. $\sqrt{25 \cdot 36}$ | 12. $\sqrt{3^2x^2y^2}$ | 18. $-\sqrt{36b^{4m}c^{2n}}$ |

Notice that $\sqrt{9 + 16}$ is not equal to $\sqrt{9} + \sqrt{16}$.

The preceding exercises illustrate

Principle XVIII

169. Rule. *The square root of a product is obtained by finding the square root of each factor separately and then taking the product of these roots. That is,*

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}.$$

In order that a factor may be a perfect square it must be a power whose exponent is *even*. Its square root is then a power of the same base whose exponent is equal to one-half the given exponent.

$$\text{Thus,} \quad \sqrt{x^6} = \sqrt{x^3 \cdot x^3} = x^3 = x^{6 \div 2}.$$

Hence to find the square root of a monomial we divide the exponent of each factor by 2.

EXERCISES

Find the following square roots :

- | | | |
|---------------------------------------|---------------------------------------|--------------------------------------|
| 1. $-\sqrt{4 a^6 b^8}$. | 6. $-\sqrt{10^4 a^4 b^4}$. | 11. $\sqrt{81 x^4 y^8 c^{10}}$. |
| 2. $-\sqrt{3^2 x^{12} y^{14}}$. | 7. $\sqrt{5^4 m^{14}}$. | 12. $\sqrt{729 a^4 y^{10} z^{14}}$. |
| 3. $\sqrt{5^2 \cdot 3^{22} t^{24}}$. | 8. $\sqrt{5^4 \cdot 3^8 \cdot 7^2}$. | 13. $-\sqrt{64 \cdot 625 a^2 b^4}$. |
| 4. $\sqrt{121 x^4 y^{12}}$. | 9. $\sqrt{3^{14} \cdot 7^{12} a^4}$. | 14. $\sqrt{256 x^{22} y^{28}}$. |
| 5. $-\sqrt{576 a^2 b^4}$. | 10. $-\sqrt{25 a^2 b^4 c^{12}}$. | 15. $\sqrt{3^{22} \cdot 5^{22}}$. |

SQUARE ROOTS OF POLYNOMIALS

170. In § 159 the square roots of certain polynomials were found by inspection. Any polynomial square may be recognized and the square root found by that method, provided no similar terms have been combined in the square.

171. The case where some terms of the square combine is illustrated by the example :

$$(x^2 + x + 1)^2 = x^4 + x^2 + 1 + 2x^3 + 2x^2 + 2x = x^4 + 2x^3 + 3x^2 + 2x + 1.$$

Study the answers to the following questions. They are needed in finding the square root of such an expression :

1. What terms must be added to a^2 to make it the square of $a + b$?

2. What terms must be added to $a^2 + 2ab + b^2$ to make it the square of $a + b + c$?

3. What terms must be added to $a^2 + 2ab + b^2 + 2ac + 2bc + c^2$ to make it the square of $a + b + c + d$.

The above answers may be summarized as follows :

When a term is added to a polynomial, the square of the resulting polynomial is equal to the square of the original, plus twice the product of the new term and the sum of the original terms, plus the square of the new term.

This is expressed by the following formula: $(a + b + c + d)^2 = a^2 + 2ab + b^2 + 2(a + b)c + c^2 + 2(a + b + c)d + d^2$.

172. Illustrative Example. Find the square root of

$$16x^6 - 24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25.$$

Solution

	a	$+ b$	$+ c + d$	
Square root	$4x^3$	$- 3x^2$	$+ 2x - 5$	
Given square	$16x^6$	$- 24x^5$	$+ 25x^4$	$- 52x^3 + 34x^2 - 20x + 25$ (1)
	$a^2 = 16x^6$			
		$- 24x^5$	$+ 25x^4$	$- 52x^3 + 34x^2 - 20x + 25$ (2)
$2ab + b^2 =$		$- 24x^5$	$+ 9x^4$	(3)
			$16x^4$	$- 52x^3 + 34x^2 - 20x + 25$ (4)
$2(a+b)c + c^2 =$		$16x^4$	$- 12x^3$	$+ 4x^2$ (5)
			$- 40x^3$	$+ 30x^2 - 20x + 25$ (6)
$2(a+b+c)d + d^2 =$			$- 40x^3$	$+ 30x^2 - 20x + 25$ (7)

Explanation. First arrange the terms according to increasing or decreasing powers of the letter involved, as in long division. The first term of the root is the square root of the first term of the polynomial. The square of this is subtracted, leaving (2).

The next term of the root, $-3x^2$, is found by dividing the first term of (2) by $2(4x^3) = 8x^3$, that is, $-24x^5 + 8x^3 = -3x^2$, corresponding to $2ab + 2a = b$.

We then subtract $2(4x^3)(-3x^2) + (-3x^2)^2 = -24x^5 + 9x^4$, corresponding to $2ab + b^2$, and this completes the subtraction of $(4x^3 - 3x^2)^2$.

Subtracting (5) completes the subtraction of $(4x^3 - 3x^2 + 2x)^2$, and subtracting (7) completes the subtraction of $(4x^3 - 3x^2 + 2x - 5)^2$.

At each step the next term of the root is found by dividing the first term of the remainder by twice the first term of the root. Thus from (4) the third term of the root is $16x^4 \div 2(4x^3) = 2x$, corresponding to $2ac \div 2a = c$.

Since the remainder is zero after the last subtraction, this shows that $(4x^3 - 3x^2 + 2x - 5)^2$ is exactly the given polynomial.

EXERCISES

Find the square roots of the following polynomials:

1. $x^4 + 2x^3 + 3x^2 + 2x + 1$.
2. $1 - 2a + 3a^2 - 2a^3 + a^4$.
3. $1 + 2b - b^2 - 2b^3 + b^4$.
4. $a^4 + 4a^3 + 6a^2 + 4a + 1$.

5. $c^4 - 4c^3 + 6c^2 - 4c + 1$.
6. $x^4 - 2x^3 + 5x^2 - 4x + 4$.
7. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.
8. $x^4 - 4x^2y + 6x^2y^2 - 4xy^3 + y^4$.
9. $a^4 + 53a^2 + 14a^3 + 28a + 4$.
10. $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1$.
11. $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1$.
12. $4x^6 - 12x^5 + 13x^4 - 14x^3 + 13x^2 - 4x + 4$.
13. $16a^6 + 24a^5 + 25a^4 + 20a^3 + 10a^2 + 4a + 1$.
14. $x^6y^6 + 2x^5y^5 + 3x^4y^4 + 4x^3y^3 + 3x^2y^2 + 2xy + 1$.
15. $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 4x^5 + 3x^6 + 2x^7 + x^8$.

SQUARE ROOTS OF NUMBERS EXPRESSED IN ARABIC FIGURES

173. The square root of a number expressed in Arabic figures may be found by the process just used for polynomials.

Illustrative Example. Find the square root of 405769.

Solution.

SQUARE ROOT	SQUARE
$a + b + c$	
$600 + 30 + 7 = 637$	405769
$a^2 = \overline{600^2}$	360000
$2ab = 36000$	45769
$b^2 = \overline{900}$	36900
$\quad\quad\quad 36900$	8869
$2(a + b)c = 8820$	
$c^2 = \overline{49}$	
$\quad\quad\quad 8869$	8869
	0

Explanation. We see that the given number is greater than the square of 600 and less than the square of 700. Hence we take 600 as the first term of the root and subtract $\overline{600^2}$ from the number.

As in the case of the polynomial the second term of the root is

SQUARE ROOTS OF NUMBERS EXPRESSED IN FIGURES 181

found by dividing the remainder by twice the first term of the root, that is $45769 \div 1200$. This gives a quotient greater than 30 and less than 40 and hence 30 is the next term of the root.

The third term of the root is found by dividing 8869 by 2×630 , that is, by $2(a + b)$. In the case of a polynomial it is sufficient to divide by $2a$, but with arithmetical numbers the third term is usually more easily found if we divide by $2(a + b)$.

The remaining parts corresponding to $(a + b)^2$ and $(a + b + c)^2$ are now computed and subtracted exactly as in the case of a polynomial.

174. The first term of the root may be found as follows:

Separate the number into groups of two digits each from the decimal point toward the left. Take the square root of the largest square contained in the last group to the left and adjoin one zero for each remaining group.

Thus in the root of 87 23 56 the first term is 900, since 81 is the largest square in the left group and there are two other groups. In the root of 7 34 86.593, the first term is 200, since 4 is the largest square contained in the left group and there are two other groups

EXERCISES

Find the square root of each of the following:

- | | | | |
|-------------|------------|--------------|--------------|
| 1. 294,849. | 5. 3481. | 9. 100,489. | 13. 357.21. |
| 2. 37,636. | 6. 7569. | 10. 265.69. | 14. 16,641. |
| 3. 872,356. | 7. 1849. | 11. 87.4225. | 15. 32,761. |
| 4. 599,076. | 8. 73,441. | 12. 170,569. | 16. 2332.89. |

175. In case the number whose square root is to be found has no figure to the left of the decimal point, the first term of the root may be found by the following rule:

Separate the number into groups of two digits from the decimal point toward the right. If necessary, add a zero to get one complete group not all zeros. Take the square root of the largest square contained in the first group which is not all zeros and prefix to it as many zeros as there are groups preceding this one, thus locating the decimal point in the root.

Thus, in the root of .03 42 the first term is .1, since 1 is the largest square in 3. Similarly in the square root of .0070 the first term is .08, since 64 is the largest square in 70 and there is one group preceding this one.

Illustrative Example. Find the square root of .06783.

Solution.

SQUARE ROOT	SQUARE
$a + b + c$	
$.2 + .06 + .0004 = .2604$.06783
$a^2 =$.04
$2 ab = .024$.02783
$b^2 = .0036$	
.0276	.0276
$2(a + b) c = .000208$.00023
$c^2 = .00000016$	
.00020816	.00020816
	.00002184

Explanation. According to the rule, .2 is the first term of the root since 4 is the largest square in 6 and there is no group preceding .06. The process is the same as in the case of an integral square, but special care is now needed in handling the decimal points, which is done exactly as in operations upon decimals in arithmetic.

For instance, in finding the third term in this example, we divide .00023 by $2(.26) = .52$ and the quotient lies between .0004 and .0005. Hence $c = .0004$.

176. Evidently the process in this example may be carried on indefinitely. .2604 is an **approximation** to the square root of .06783. In fact, the square of .2604 differs from .06783 by only .00002184. .260 is the nearest approximation using three decimal places. If the fourth figure were 5, or any digit greater than 5, then .261 would be the nearest approximation using three decimal places. Hence, four places must be found in order to be sure of the nearest approximation to three places.

EXERCISES

Find the square roots of the following, correct to two decimal places:

1. 387.	7. 2.	13. .02.
2. 5276.	8. 3.	14. .003.
3. 2.92.	9. 5.	15. .5.
4. 27.29.	10. 7.	16. .005.
5. 51.	11. 8.	17. .307.
6. 3.824.	12. 11.	18. 200.002.

SQUARE ROOTS OF FRACTIONS

177. A fraction is squared by squaring its numerator and its denominator separately, since $\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$. Hence, to extract the square root of a fraction, we find the square root of its numerator and its denominator separately.

E.g. $\sqrt{\frac{1}{9}} = \frac{1}{3}$, since $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

However, in approximating the square root of a fraction whose denominator is not a perfect square, the fraction may be reduced to a decimal before the root is approximated.

E.g. $\sqrt{\frac{1}{3}} = \sqrt{.666 \dots} = .8165 \dots$

It is also sometimes convenient to make the denominator of a fraction a perfect square before approximating the root.

E.g. $\sqrt{\frac{1}{3}} = \sqrt{\frac{4}{12}} = \sqrt{\frac{1}{3} \cdot 6} = \sqrt{\frac{1}{3}} \cdot \sqrt{6} = \frac{1}{3} \sqrt{6} = \frac{2.4495}{3} = .8165 \dots$

Note that Principle XVIII is used in taking the step,

$$\sqrt{\frac{1}{3} \cdot 6} = \sqrt{\frac{1}{3}} \cdot \sqrt{6}.$$

It is clear that *any fraction can be changed into an equal fraction whose denominator is a perfect square by multiplying numerator and denominator by the proper number.*

E.g. $\frac{1}{2} = \frac{4}{8}$, $\frac{2}{3} = \frac{8}{12}$, $\frac{1}{4} = \frac{9}{36}$, etc.

178. Example. Find the square root of $\frac{3}{5}$ by finding the roots of 2 and 3 separately and then dividing to reduce to a decimal.

Compare this with the two methods given above and show in what respect they are simpler.

179. If it is required to approximate the value of $\frac{1}{\sqrt{5}}$, the simplest method is as follows:

$$\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{\sqrt{5}}{5}. \quad \text{Similarly, } \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{3\sqrt{5}}{5}.$$

Note that $\sqrt{5} \sqrt{5} = 5$ by Principle XVIII read in reverse order.

$$\text{Thus, } \sqrt{5} \sqrt{5} = \sqrt{25} = 5.$$

EXERCISES

State three rules for finding the square root of a fraction.

Find approximately correct to two decimal places the following square roots, using the method which involves the least computation.

- | | | | |
|----------------------------|-----------------------------|----------------------------|-----------------------------|
| 1. $\sqrt{\frac{3}{8}}$. | 7. $\sqrt{\frac{31}{8}}$. | 13. $\sqrt{\frac{1}{2}}$. | 18. $\frac{5}{\sqrt{13}}$. |
| 2. $\sqrt{\frac{1}{8}}$. | 8. $\sqrt{\frac{42}{11}}$. | 14. $\sqrt{\frac{3}{7}}$. | 19. $\frac{3}{\sqrt{7}}$. |
| 3. $\sqrt{\frac{5}{7}}$. | 9. $\sqrt{\frac{5}{7}}$. | 15. $\sqrt{\frac{1}{7}}$. | 20. $\frac{7}{\sqrt{17}}$. |
| 4. $\sqrt{\frac{11}{8}}$. | 10. $\sqrt{\frac{5}{8}}$. | 16. $\sqrt{\frac{2}{6}}$. | 21. $\sqrt{\frac{8}{3}}$. |
| 5. $\sqrt{\frac{8}{3}}$. | 11. $\sqrt{\frac{25}{4}}$. | 17. $\frac{1}{\sqrt{5}}$. | |
| 6. $\sqrt{\frac{1}{8}}$. | 12. $\sqrt{\frac{16}{7}}$. | | |

SIMPLIFYING RADICALS

180. Principle XVIII may also be used to advantage in approximating the square roots of certain integral numbers.

E.g. Suppose $\sqrt{2}$ has been computed, and $\sqrt{8}$ is desired. It is unnecessary to compute the $\sqrt{8}$ directly, for by XVIII,

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$$

This sort of simplification is possible whenever the number under the radical sign can be resolved into two factors, one of which is a perfect square.

E.g. Suppose $\sqrt{5}$ to have been computed, then

$$\sqrt{125} = \sqrt{25 \cdot 5} = \sqrt{25} \cdot \sqrt{5} = 5\sqrt{5}.$$

In like manner, $\sqrt{a^5b^3}$ may be written

$$\sqrt{a^4b^2 \cdot ab} = \sqrt{a^4b^2} \cdot \sqrt{ab} = a^2b\sqrt{ab}.$$

181. Definition. An expression in one of the forms $\sqrt{a^2b}$, $\frac{a}{\sqrt{b}}$, $\sqrt{\frac{a}{b}}$, is said to be **simplified** when it is reduced so that no radical occurs in a denominator and when the number under the radical sign is in the integral form and contains no factor which is a perfect square.

Such radical expressions may always be simplified by Principle XVIII.

E.g.

$$\sqrt{125} = \sqrt{25 \cdot 5} = 5\sqrt{5}$$

$$\sqrt{a^5b^3} = \sqrt{a^4b^2 \cdot ab} = a^2b\sqrt{ab}$$

$$\sqrt{\frac{1}{3}} = \sqrt{\frac{1}{3} \cdot 5} = \frac{1}{\sqrt{3}}\sqrt{5}$$

$$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{3}}{3}$$

EXERCISES

Given $\sqrt{2} = 1.4142$, $\sqrt{3} = 1.7321$, $\sqrt{5} = 2.2361$, compute the following, correct to three places of decimals, without further extraction of roots:

1. $\sqrt{80}$.

6. $\sqrt{2 \cdot 3}$.

11. $\sqrt{27} + \sqrt{\frac{1}{3}}$.

2. $\sqrt{\frac{1}{3}}$.

7. $\sqrt{72}$.

12. $\sqrt{45} + \sqrt{\frac{1}{3}}$.

3. $\sqrt{\frac{1}{3}}$.

8. $\sqrt{98}$.

13. $\sqrt{50} - \sqrt{\frac{1}{2}} + \sqrt{8}$.

4. $\sqrt{48}$.

9. $\sqrt{363}$.

14. $\sqrt{48} + \sqrt{75} - \sqrt{3}$.

5. $\sqrt{75}$.

10. $\sqrt{125}$.

15. $\sqrt{32} + \sqrt{72} - \sqrt{18}$.

Simplify the following:

16. $\sqrt{32 a^3 b}$. $\sqrt[3]{19. \sqrt{45 x^2 y^3 b^3}}$. 22. $\sqrt{500 x^2 a^3 b}$.
 $\sqrt[3]{17. \sqrt{81 x^3 b^3}}$. $\sqrt[3]{20. \sqrt{63 b c^3 a^4}}$. 23. $\sqrt{3 x^2 + 6 xy + 3 y^2}$.
 $\sqrt[3]{18. \sqrt{50 a^3 b^4 c^3}}$. 21. $\sqrt{900 ab^4 c^5}$. 24. $\sqrt{8 x^2 - 12 y^2}$.
 25. $\sqrt{32 a^2 - 64 ab + 32 b^2}$. 26. $\sqrt{125 x^2 + 250 xy + 125 y^2}$.

27. Find approximately to two decimal places the sides of a square whose area is 120.

28. Approximate to two decimals the side of a square having an area equal to that of a rectangle whose sides are 15 and 20.

29. How many rods of fence are required to fence a square piece of land containing 50 acres, each acre containing 160 square rods?

30. A square checkerboard has an area of 324 square inches. What are its dimensions?

In adding or subtracting expressions containing radicals it is always best to first reduce each radical expression to its simplest form, since this often gives opportunity to combine terms which are similar with respect to some radical expression.

Ex. 1. $\sqrt{32} + \sqrt{72} - \sqrt{18} = 4\sqrt{2} + 6\sqrt{2} - 3\sqrt{2} = 7\sqrt{2}$ by Principles XVIII and I.

$$\begin{aligned} \text{Ex. 2. } \sqrt{\frac{1}{3}} + \sqrt{12} - \sqrt{\frac{4}{3}} &= \frac{1}{3}\sqrt{3} + 2\sqrt{3} - \frac{2}{3}\sqrt{3} \\ &= (\frac{1}{3} + 2 - \frac{2}{3})\sqrt{3} = 1\frac{2}{3}\sqrt{3}. \end{aligned}$$

EXERCISES

Simplify each of the following as far as possible without approximating roots.

1. $\sqrt{27} + 2\sqrt{48} - 3\sqrt{75}$. 3. $3\sqrt{432} - 4\sqrt{3} + \sqrt{147}$.
 $\sqrt[3]{2. \sqrt{20} + \sqrt{125} - \sqrt{180}}$. 4. $\sqrt[3]{3\sqrt{2450} - 25\sqrt{2} + 4\sqrt{13122}}$.

5. $3y^3\sqrt{x^2z} + 2\sqrt{x^5z^3} - yz^4\sqrt{\frac{xz}{y^3}}$.
6. $\sqrt{4x^3y} + \sqrt{25xy^3} - x\sqrt{xy}$.
7. $\sqrt{ax^2 - bx^3} + \sqrt{4ar^2s^3 - 4br^2s^3}$.
8. $4\sqrt{\frac{3}{4}} - \frac{3}{4}\sqrt{\frac{3}{16}} - 3\sqrt{27}$.
9. $2\sqrt{\frac{5}{8}} + \sqrt{60} + \sqrt{\frac{3}{8}}$.
10. $5\sqrt{3} - 2\sqrt{48} + 7\sqrt{108}$.
11. $\sqrt{a^3 - a^2b} - \sqrt{ab^3 - b^3} - \sqrt{(a+b)(a^2 - b^2)}$.
12. $\sqrt{a} + 3\sqrt{2a} - 2\sqrt{3a} + \sqrt{4a} - \sqrt{8a} + \sqrt{12a}$.
13. $\sqrt{x^3 + 2x^2y + xy^3} - \sqrt{x^3 - 2x^2y + xy^3} - \sqrt{4xy^2}$.
14. $\sqrt{r-s} + \sqrt{16r-16s} + \sqrt{rt^2-st^2} - \sqrt{9(r-s)}$.
15. $\sqrt{(m-n)^2a} + \sqrt{(m+n)^2a} - \sqrt{am^3} + \sqrt{a(1-m)^2} - \sqrt{a}$.
16. $\sqrt{32x^2y^4} + \sqrt{162x^2y^4} - \sqrt{512x^2y^4} + \sqrt{1250x^2y^4}$.

EQUATIONS SOLVED BY SQUARE ROOTS

182. Since $2^2 = 4$ and also $(-2)^2 = 4$, it follows that the equation $x^2 = 4$ has two roots, namely $x=2$ and $x=-2$. These are usually written $x = \pm 2$.

This solution is obtained by taking the square root of both sides, which is equivalent to dividing both sides by the same number.

This operation may now be added to those enumerated in Principle VI for the solution of equations.

EXERCISES

Find all roots of the following equations:

- | | | |
|-------------------------|------------------------------|------------------------|
| 1. $x^2 = 9$. | 6. $x^2 = 64am^6n^4a^3$. | 11. $x^2 = 98$. |
| 2. $x^2 = 25$. | 7. $x^2 = 36r^2s^4$. | 12. $x^2 = 80$. |
| 3. $x^2 = 16a^2$. | 8. $x^2 = 81s^3r^6$. | 13. $x^2 = 175$. |
| 4. $x^2 = 49b^2$. | 9. $x^2 = 625a^4b^8$. | 14. $x^2 = 49a^5$. |
| 5. $x^2 = 81a^4b^3$. | 10. $x^2 = 72$. | 15. $x^2 = 36a^3b^5$. |
| 16. $x^2 = 25(a-b)^2$. | 18. $x^2 = 200(a+b)^3$. | |
| 17. $x^2 = 50(a+b)$. | 19. $x^2 = 1250(a-b)^3c^5$. | |

DRILL EXERCISES

Using the formulas for $(a \pm b)^2$, obtain the squares of the following:

1. $(a + b) + (c - d)$.
2. $(a + 3) - (b + c)$.
3. $(3a - 2b) + 5$.
4. $7x - (4r - s)$.
5. $(m^2 - 3) - 2(m^2 + n)$.
6. $3(2 + y) - 2(3 + x)$.

Factor the following:

7. $(2 - x)^2 - 2(2 - x)(x - 1) + (x - 1)^2$.
8. $(2 + y)^2 + 2(2 + y)(1 + x) + (1 + x)^2$.
9. $(3a - 2b)^2 - 10(3a - 2b) + 25$.
10. $(6a - b)^2 + (2a + 1)^2 - 2(6a - b)(2a + 1)$.
11. $25(a + b)^2 + 50(a + b)(a - b) + 25(a - b)^2$.
12. $x^2 + 12x(a + b + c) + 36(a + b + c)^2$.
13. $49(m - 3)^2 + 36(m + 1)^2 - 84(m - 3)(m + 1)$.
14. $16(x - y)^2 - 16(x - y)(x + y) + 4(x + y)^2$.
15. $-30(a + b)(a - b)^2 + 25(a - b)^4 + 9(a + b)^2$.

Using the formula for $(a + b)(a - b)$, write out the following products:

16. $[a + b + (c - d)][a + b - (c - d)]$.
17. $[x + y + (u + v)][x + y - (u + v)]$.
18. $[4x - (a - 2b)][4x + (a - 2b)]$.
19. $[a + 2b - (x - y^2)][a + 2b + (x - y^2)]$.
20. $(11b^2x - 3bx^3)(11b^2x + 3bx^3)$.

Factor the following:

21. $a^2 + 4ab + 4b^2 - (x^2 - 2xy + y^2)$.
22. $(3x - 2)^2 - (4x^2 + 9y^2 - 12xy)$.
23. $x^2 + 4xy + 4y^2 - (a^2 + 2ab + b^2)$.
24. $16x^4y^2 - (4x^2 + 9y^2 + 12xy)$.
25. $(a + b)^2 - (4a^2 + 9b^2 - 12ab)$.

APPLICATIONS OF SQUARE ROOT

183. Some of the most interesting and useful applications of the square root process are concerned with the sides and areas of triangles.

The fact that the sum of the squares on the two sides of a right triangle equals the square on the hypotenuse was used in Chapter X. (Pythagorean Proposition, page 172.)

If a and b are the lengths of the sides, and c the length of the hypotenuse, all measured in the same unit, this proposition says:

$$c^2 = a^2 + b^2. \quad (1)$$

$$\text{Hence, by } S, \quad a^2 = c^2 - b^2, \quad (2)$$

$$\text{and} \quad b^2 = c^2 - a^2. \quad (3)$$

Taking the square root of both sides in each of these equations,

$$c = \sqrt{a^2 + b^2}. \quad (4)$$

$$a = \sqrt{c^2 - b^2}. \quad (5)$$

$$b = \sqrt{c^2 - a^2}. \quad (6)$$

The negative square root is omitted here, as a negative length cannot apply to the side of a triangle. By these formulas, if any two sides of a right triangle are given, the other may be found.

E.g. if

$$a = 4, b = 3, \text{ then, by (4),}$$

$$c = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$

If

$$c = 5, b = 3, \text{ then, by (5),}$$

$$a = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4.$$

If

$$c = 5, a = 4, \text{ then, by (6),}$$

$$b = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3.$$

Illustrative Problem. If the two sides of a right triangle are 8 and 12, find the hypotenuse correct to two decimal places.

Solution. We have $c = \sqrt{a^2 + b^2} = \sqrt{64 + 144} = \sqrt{208},$

$$\sqrt{208} = \sqrt{16 \cdot 13} = \sqrt{16} \cdot \sqrt{13} = 4\sqrt{13} = 4(3.605) = 14.420.$$

PROBLEMS

In solving the following problems, simplify each expression under the radical sign before extracting the root. Find all results correct to two decimal places.

1. The sides about the right angle of a right triangle are each 15 inches. Find the hypotenuse.

2. The hypotenuse of a right triangle is 9 inches and one of the sides is 6 inches. Find the other side.

3. The hypotenuse of a right triangle is 25 feet and one of the sides is 15 feet. Find the other side.

4. The hypotenuse of a right triangle is 7 rods and one of the sides is 5 rods. Find the other side.

5. The hypotenuse of a right triangle is 12 inches and the two sides are equal. Find their length.

Let s be the length of one of the equal sides.

$$\text{Then} \quad s^2 + s^2 = 144.$$

$$2s^2 = 144.$$

$$s^2 = 72.$$

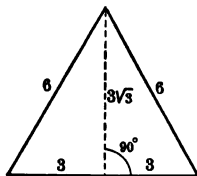
$$s = \sqrt{72} = 6\sqrt{2} = 6 \times 1.414 = 8.484.$$

6. The hypotenuse of a right triangle is 30 feet and the sides are equal. Find their length.

7. The hypotenuse of a right triangle is h and the sides are equal. Find their length. Solve Exs. 5 and 6 by means of the formula here obtained.

8. The diagonal of a square is 8 feet. Find its area.

9. The diagonal of a square is d . Find an expression in terms of d representing its area.



10. The side of an equilateral triangle is 6 inches. Find the altitude.

A line drawn from a vertex of an equilateral triangle perpendicular to the base meets the base at its middle point. Hence this problem becomes: the hypotenuse of a right triangle is 6 and one side is 3. Find the remaining side.

11. The side of an equilateral triangle is 10. Find the altitude.

12. The side of an equilateral triangle is s . Find the altitude.

This is equivalent to finding a side of a right triangle whose hypotenuse is s , the other side being $\frac{s}{2}$. Let h equal altitude.

$$\begin{aligned}\text{Then} \quad h &= \sqrt{s^2 - \left(\frac{s}{2}\right)^2} = \sqrt{s^2 - \frac{s^2}{4}} \\ &= \sqrt{\frac{4s^2 - s^2}{4}} = \sqrt{\frac{3s^2}{4}} = \sqrt{\frac{s^2}{4}} \cdot \sqrt{3} \\ &= \sqrt{\frac{s^2}{4}} \cdot \sqrt{3} = \frac{s}{2} \sqrt{3}.\end{aligned}$$

This formula gives the altitude of any equilateral triangle in terms of the side. By means of this formula solve Exs. 11 and 12.

13. Find the altitude of an equilateral triangle whose side is $4\frac{1}{2}$. Substitute in the formula under Ex. 12.

14. Find the area of an equilateral triangle whose side is 5.

Since the area of a triangle is $\frac{1}{2}$ the product of the base and altitude, we first find the altitude by means of the formula under Ex. 12, and then multiply by $\frac{1}{2}$ the base.

15. Find the area of the equilateral triangle whose side is s . Show the result to be $\frac{s^2}{4} \sqrt{3}$.

16. If the area of an equilateral triangle is 16 square inches, find the length of the side.

Let s equal the length of the side. Then by the formula derived in Ex. 15, we have $16 = \frac{s^2}{4} \sqrt{3}$.

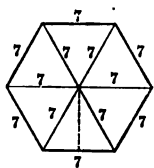
$$\text{Hence (§§ 177-180), } s^2 = \frac{64}{\sqrt{3}} = \frac{64}{3} \sqrt{3} = 21.33 \times 1.732.$$

17. The area of an equilateral triangle is 50 square inches. Find its side and altitude.

18. The area of an equilateral triangle is a square inches. Find the side.

Solve the equation $a = \frac{s^2}{4} \sqrt{3}$ for s , and simplify the expression, finding $s^2 = \frac{4a}{\sqrt{3}}$, and $s = \sqrt{\frac{4a\sqrt{3}}{3}} = \frac{2}{3} \sqrt{3a\sqrt{3}}$.

19. The area of an equilateral triangle is 240 square inches. Find its side. (Substitute in the formula obtained in Ex. 18.)



20. Find the area of a regular hexagon whose side is 7.

A regular hexagon is composed of 6 equal equilateral triangles, whose sides are each equal to the side of the hexagon (see figure). Hence this problem may be solved by finding the area of an equilateral triangle whose side is 7, and multiplying the result by 6.

21. Find the area of a regular hexagon whose side is s . Solve Ex. 20 by substituting in the formula obtained here.

22. The area of a regular hexagon is 108 square inches. Find its side.

If the area of the hexagon is 108 square inches, the area of one of the equilateral triangles is 18 square inches. Hence this problem can be solved like Ex. 18.

23. The area of a regular hexagon is a square inches. Find its side. Solve Ex. 22 by substituting in the formula obtained here.

$$\text{Ans. } s = \frac{1}{3} \sqrt{2a\sqrt{3}}.$$

24. Find the radius of a circle whose area is 9 square inches.

The area of a circle is found by squaring the radius and multiplying by 3.1416. The number 3.1416 is approximately the quotient obtained by dividing the length of the circumference by the diameter of the circle. This quotient is represented by the Greek letter π (pronounced pi). In this chapter we use $3\frac{1}{2}$ as an approximation to π . This differs from the real value of π by less than .0013, and hence is accurate enough for most purposes. If a represents the area of a circle, the above rule may be written

$$a = \pi r^2.$$

Hence if $a = 9$,
$$r^2 = \frac{9}{\pi} = \frac{9}{3\frac{1}{7}} = \frac{63}{22} = 2.863,$$

and

$$r = \sqrt{2.863}.$$

25. Find the radius of a circle whose area is 68 square feet.

26. Find the radius of a circle whose area is a square feet.

We have

$$a = \pi r^2, \text{ or } r^2 = \frac{a}{\pi}.$$

Hence

$$r = \sqrt{\frac{a}{\pi}} = \sqrt{\frac{a\pi}{\pi^2}} = \frac{1}{\pi} \sqrt{a\pi}.$$

In problems stated in terms of letters, the results, of course, cannot be reduced to a decimal. In such formulas it is best not to replace the letter π by any of its approximations.

FURTHER OPERATIONS ON RADICALS

184. The radical sign is used to indicate other roots than square roots by means of an *index* figure.

Thus, the *cube root* of 8, or one of its three equal factors, is written $\sqrt[3]{8} = 2$. The *fourth root* of 16, or one of its four equal factors, is written $\sqrt[4]{16} = 2$.

185. **Definitions.** Any expression which contains an indicated root is called a **radical expression**.

Integers and fractions of the form $\frac{m}{n}$ where m and n are integers are called **rational numbers**.

E.g. $2 + \sqrt[3]{5}$ is a radical expression. $5, \frac{1}{2}, \sqrt[4]{16}$, are rational numbers.

186. An expression which consists of a rational number under a radical sign, or one which can be reduced to this form, is called a **surd**, provided the whole expression is not reducible to a rational number.

E.g. $\sqrt{2}$ is a surd since it cannot be reduced to a rational number. $\sqrt[3]{4}, \sqrt[5]{3}$ are surds for the same reason. $\sqrt{9}$ is not a surd since $\sqrt{9} = 3$. $\sqrt{2} + \sqrt{2}$ is not a surd since $2 + \sqrt{2}$ is not a rational number.

187. Surd expressions containing indicated square roots only, and in which no number is under more than one radical sign, are called **quadratic surds**.

E.g. $\sqrt{2} + \sqrt{3}$, $3 + \sqrt{5}$, $\frac{3}{\sqrt{7} - \sqrt{5}}$, are quadratic surds, while $\sqrt{\sqrt{2}}$ and $\sqrt[3]{4}$ are not quadratic surds.

The following operations upon radicals deal only with quadratic surds.

MULTIPLICATION OF QUADRATIC SURDS

188. By Principle XVIII $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

This equation read in the reverse order gives a rule for multiplying quadratic surds.

E.g. $\sqrt{2} \cdot \sqrt{8} = \sqrt{2 \cdot 8} = \sqrt{16} = 4.$

$$\sqrt{3} \cdot \sqrt{15} = \sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}.$$

EXERCISES

Make a rule for multiplying quadratic surds.

Perform the following indicated operations and reduce each result to the simplest form.

- | | | |
|--------------------------------|-------------------------------------------|---------------------------------------------------|
| 1. $\sqrt{5} \cdot \sqrt{7}.$ | 4. $\sqrt{x^3} \cdot \sqrt{x^5}.$ | 7. $\sqrt{7} \cdot \sqrt{\frac{1}{7}}.$ |
| 2. $\sqrt{6} \cdot \sqrt{12}.$ | 5. $\sqrt{a^2b} \cdot \sqrt{ab^2}.$ | 8. $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{1}{3}}.$ |
| 3. $\sqrt{18} \cdot \sqrt{8}.$ | 6. $\sqrt{rs^2t} \cdot \sqrt{r^3s^2t^3}.$ | 9. $\sqrt{\frac{1}{5}} \cdot \sqrt{\frac{4}{5}}.$ |
| 10. $(\sqrt{2} + \sqrt{3})^2.$ | | |

Solution. $(\sqrt{2} + \sqrt{3})(\sqrt{2} + \sqrt{3}) = (\sqrt{2})^2 + 2\sqrt{2}\sqrt{3} + (\sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}.$

- | | |
|---------------------------------------------------|-------------------------------------------------------|
| 11. $(\sqrt{2} - \sqrt{3})^2.$ | 16. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}).$ |
| 12. $(1 + \sqrt{2})^2.$ | 17. $\sqrt{2}(2\sqrt{3} + 3\sqrt{8} - 5\sqrt{6}).$ |
| 13. $(2 - \sqrt{3})^2.$ | 18. $\sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c}).$ |
| 14. $(1 + \sqrt{2})(1 - \sqrt{2}).$ | 19. $(3\sqrt{5} - 2\sqrt{3})(3\sqrt{5} + 2\sqrt{3}).$ |
| 15. $(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5}).$ | 20. $(1 + \sqrt{2})(1 - \sqrt{3} + \sqrt{5}).$ |

DIVISION OF QUADRATIC SURDS

189. When a divisor is a quadratic surd, it is convenient to indicate the division in the form of a fraction and then to reduce the denominators to the rational form as in the following example:

$$\frac{\sqrt{5}}{\sqrt{5}-\sqrt{2}} = \frac{\sqrt{5}(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})} = \frac{5+\sqrt{10}}{5-2} = \frac{5+\sqrt{10}}{3}.$$

If it is desired to compute the approximate value of $\sqrt{5} + (\sqrt{5} - \sqrt{2})$, it obviously requires less numerical work to use the form $\frac{5+\sqrt{10}}{3}$ rather than $\frac{\sqrt{5}}{\sqrt{5}-\sqrt{2}}$; since the latter involves the extraction of two square roots and a long division, while the former requires the extraction of only one root and a short division.

190. The process indicated in the above example is called **rationalizing the denominator**, and the factor by which the terms of the fraction are multiplied is called the **rationalizing factor**.

If the denominator is of the form \sqrt{x} or $a\sqrt{x}$, then \sqrt{x} is the rationalizing factor, since $\sqrt{x} \cdot \sqrt{x} = x$.

If the denominator is of the form $\sqrt{x} + \sqrt{y}$, then $\sqrt{x} - \sqrt{y}$ is the rationalizing factor, since $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = x - y$.

Give a rationalizing factor of each of the following:

- (1) $\sqrt{3x}$, (2) $\sqrt{2x^3b}$, (3) $\sqrt{8b^3c^3}$, (4) $\sqrt{a} + \sqrt{b}$,
 (5) $\sqrt{a} - \sqrt{b}$, (6) $\sqrt{3a} + \sqrt{2b^3}$, (7) $\sqrt{7} - \sqrt{27}$.

191. To rationalize the denominator of any fraction, it is first necessary to find an expression which multiplied by the denominator of the fraction gives a rational product, and then to multiply both terms of the fraction by this expression.

EXERCISES

Rationalize the denominators of each of the following:

1. $\frac{1}{\sqrt{x}}$

8. $\frac{3\sqrt{6}+9\sqrt{2}}{3\sqrt{2}}$

15. $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$

2. $\frac{x+1}{\sqrt{x}}$

9. $\frac{1}{\sqrt{2}-1}$

16. $\frac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

3. $\frac{1}{\sqrt{x+1}}$

10. $\frac{1}{\sqrt{5}-\sqrt{3}}$

17. $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

4. $\frac{\sqrt{x-1}}{\sqrt{x+1}}$

11. $\frac{1}{\sqrt{3}+\sqrt{5}}$

18. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

5. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{xy}}$

12. $\frac{2}{\sqrt{7}-\sqrt{3}}$

19. $\frac{1}{a\sqrt{x}+b\sqrt{y}}$

6. $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}}$

13. $\frac{2}{\sqrt{a}-1}$

20. $\frac{a+b}{a\sqrt{x}-b\sqrt{y}}$

7. $\frac{\sqrt{8}-\sqrt{3}}{\sqrt{3}}$

14. $\frac{1}{\sqrt{a}+\sqrt{b}}$

EQUATIONS INVOLVING RADICALS

192. Illustrative Example. Solve for x the equation:

$$\sqrt{x-5} + \sqrt{x+1} = 3 \quad (1)$$

$$\text{By } S|\sqrt{x+1} \quad \sqrt{x-5} = 3 - \sqrt{x+1} \quad (2)$$

$$\text{Squaring both sides} \quad x-5 = 9 - 6\sqrt{x+1} + x+1 \quad (3)$$

$$\text{Simplifying} \quad 2\sqrt{x+1} = 5 \quad (4)$$

$$\text{Squaring both sides} \quad 4(x+1) = 25 \quad (5)$$

$$\text{Hence} \quad x = 5\frac{1}{4} \quad (6)$$

$$\text{Check. Substitute} \quad x = 5\frac{1}{4} \text{ in (1)}$$

If two radicals are involved or one radical and rational terms, it is best to get a radical alone on one side of the equation before squaring as in (2) and (4). Note that squaring both members of an equation is equivalent to multiplying both sides by the same number.

EXERCISES

Solve the following equations and check each result.

1. $\sqrt{x-5}=3$.
2. $x-2=\sqrt{x^2-4}$.
3. $\sqrt{x+6}=3\sqrt{x-2}$.
4. $\sqrt{x^2-2x+8}=x-4$.
5. $\sqrt{x^2+7x-4}=\sqrt{x^2+8x-5}$.
6. $\sqrt{x^2+5}+x=5$.
7. $\sqrt{4x^2-7}+2x=7$.
8. $\sqrt{x+1}-\sqrt{x-7}=2$.
9. $\sqrt{y+4}+\sqrt{y-1}=5$.
10. $\sqrt{x+2}=\sqrt{x+16}$.
11. $5-\sqrt{x}=\sqrt{x+5}$.
12. $\sqrt{x+7}=1+\sqrt{x+2}$.
13. $\frac{3(\sqrt{x}-2)}{\sqrt{x}-1}=\frac{\sqrt{x}+1}{\sqrt{x}+2}$.
14. $\frac{\sqrt{x-2}}{\sqrt{x-4}}=\frac{\sqrt{x+1}}{\sqrt{x-3}}$.
15. $\frac{1}{\sqrt{x}+1}+\frac{1}{\sqrt{x}-1}=\frac{4}{x-1}$.
16. $\frac{2-\sqrt{x}}{1+\sqrt{x}}=\frac{3+\sqrt{x}}{3-\sqrt{x}}$.

Suggestions. In Ex. 13 clear of fractions first; in 14 square both members and then clear of fractions; in 15 rationalize denominators.

REVIEW QUESTIONS

1. State Principle XVIII. Show by use of this principle how to find $\sqrt{28}$ having given $\sqrt{7}=2.696$.

2. Show how the value of the following may be approximated by finding only one square root.

$$5\sqrt{20}+2\sqrt{45}-3\sqrt{80}+2\sqrt{\frac{1}{5}}$$

3. Write the square of $a+b+c+d$ in such a form as to derive from it the rule for finding the square root of a polynomial.

4. When is a radical expression said to be simplified? How is Principle XVIII used for this purpose?

5. Give examples of rational expressions, of surds, of quadratic surds. When is an expression said to be rationalized?

6. What principle is used in multiplying two quadratic surds? Find the product $(1+\sqrt{2})(2+3\sqrt{2})$.

7. How is division by a quadratic surd carried out? Simplify $2\sqrt{3}+(\sqrt{2}-\sqrt{3})$; also $3\sqrt{5}+(\sqrt{5}-\sqrt{2})$.

8. In solving an equation containing radicals, how are they removed (1) when only one radical expression is involved ; (2) when two such expressions are involved ?

9. State Principle XVIII in symbols and thus complete your list. Put them all on a single sheet for handy reference.

DRILL EXERCISES

1. Divide $a^3 - 12a^2 + 27a + 40$ by $a - 5$.
2. Divide $x^5 - 5x^4y + 11x^3y^2 - 14x^2y^3 + 9xy^4 - 2y^5$ by $x^2 - 3xy + 2y^2$.
3. Divide $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.
4. Divide $a^3 + 5a^2 - 2a - 24$ by $a^2 + 7a + 12$.
5. Divide $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ by $a^2 - 2ab + b^2$.
6. Divide $x^5 - 5x^3y^2 - 5x^2y^3 + y^5$ by $x^2 - 3xy + y^2$.

Factor:

- | | | |
|---------------------------------------------------------|-------------------------------|----------------------|
| 7. $a^3 + 8$. | 13. $1 - 64x^3$. | 19. $1 + 125x^6$. |
| 8. $27a^3 + b^3$. | 14. $w^6 + 27a^6$. | 20. $27x^6 - 1$. |
| 9. $b^3 - 27$. | 15. $w^6 - 8a^3$. | 21. $1 - 8x^3y^3$. |
| 10. $8a^3 - b^3$. | 16. $27a^3 - 8b^3$. | 22. $1 + 8x^3y^3$. |
| 11. $1 + 64x^3$. | 17. $x^6 + y^6$. | 23. $8x^3 + 27y^3$. |
| 12. $a^6 - b^6$. | 18. $125a^3 + b^3$. | 24. $8x^3 - 27y^3$. |
| 25. $c^4 - 31c^2 + 220$. | 27. $26 + 39n - 22m - 33mn$. | |
| 26. $ac + d^5a - b^4c - b^4d^5$. | 28. $12x^2 + 11x - 56$. | |
| 29. $a^2 + 4ab + 4b^2 - (a^2 - 4ab + 4b^2)$. | | |
| 30. $(3x - 1)^2 - (x^2 + 4y^2 - 4xy)$. | | |
| 31. $(x + 3y)^2 + (x - 2y)^2 + 2(x + 3y)(x - 2y)$. | | |
| 32. $16(a + b)^2 - 8(a - b)(a + b) + (a - b)^2$. | | |
| 33. $256x^2 - (49x^2 + 4y^4 - 28xy^2)$. | | |
| 34. $(2x - a)^2 + 100(a - 3x)^2 + 20(2x - a)(a - 3x)$. | | |
| 35. $-48(a - b)(a + b) + 36(a - b)^2 + 16(a + b)^2$. | | |

CHAPTER XII

QUADRATIC EQUATIONS

193. Equations of the form $x^2 + ax + b = 0$ have already been solved in cases where the left members could be factored by inspection. However, in a case like $x^2 + 5x + 3 = 0$, the factors of the left member cannot be found by any method thus far studied. It is therefore necessary to consider other methods for solving equations of this type.

194. As a preliminary step let us consider again the properties of a trinomial square.

How many terms of such a trinomial must be squares? How is the remaining term related to these? What term must be added to $a^2 + 2ab$ in order to make this a trinomial square?

The process of finding this third term is called **completing the square**. Since $2ab$ must be twice the product of the square roots of the squared terms, it follows that *b of the missing term may be found by dividing $2ab$ by twice a .*

EXERCISES

Complete the trinomial square in each of the following:

- | | | |
|------------------|-------------------|-----------------------|
| 1. $x^2 + 2x$ / | 4. $x^2 + 8x$ / 6 | 7. $(3x)^2 + 2(3x)$. |
| 2. $x^2 + 4x$ 1/ | 5. $x^2 + 3x$. | 8. $(2x)^2 + 4(2x)$. |
| 3. $x^2 + 6x$ 9 | 6. $x^2 + 5x$. | 9. $16x^2 + 2(4x)$. |

10. How do you complete the square in $a^2 - 2ab$? Is the rule different in this case?

11. Complete the square in each of the above exercises, first replacing the sign $+$ by $-$.

195. Solution of a quadratic by completing the square.**Ex. 1.** Solve the equation :

$$x^2 + 6x + 4 = 0. \quad (1)$$

By $S \mid 4$,

$$x^2 + 6x = -4. \quad (2)$$

Adding 3^2 to both members to complete the square,

$$x^2 + 6x + 3^2 = 3^2 - 4 = 5. \quad (3)$$

Taking square root of both sides, $x + 3 = \pm \sqrt{5}$.

(4)

Hence

$$x = -3 + \sqrt{5} = -3 + 2.24 = -.76,$$

and

$$x = -3 - \sqrt{5} = -3 - 2.24 = -5.24.$$

Ex 2. Solve the equation :

$$x^2 - 12x + 42 = 56. \quad (1)$$

By S ,

$$x^2 - 12x = 14. \quad (2)$$

Completing the square, $x^2 - 12x + 36 = 14 + 36 = 50$.

(3)

Taking square roots,

$$x - 6 = \pm \sqrt{50} = \pm 5\sqrt{2}. \quad (4)$$

By A ,

$$x = 6 \pm 7.071. \quad (5)$$

Hence

$$x = 6 + 7.071 = 13.071,$$

and also

$$x = 6 - 7.071 = -1.071.$$

This process is called solving the quadratic equation by completing the square.

Make a rule for solving a quadratic equation by this process.

EXERCISES

In solving the following quadratic equations the result may in each case be reduced so that the number remaining under the radical sign shall be 2, 3, or 5. (§ 180.) Use $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, $\sqrt{5}=2.236$.

1. $x^2 - 4x = 8$.

6. $x^2 - 12x = 12$.

11. $8 = x^2 + 4x$.

2. $x^2 = 3 - 6x$.

7. $x^2 - 8x = -14$.

12. $23 - 6x = x^2$.

3. $4x = 16 - x^2$.

8. $x^2 = 2x + 1$.

13. $7 + 2x = x^2$.

4. $x^2 + 6x = 9$.

9. $x^2 - 4x = 16$.

14. $25 - x^2 = 5x$.

5. $x^2 + 6x = 11$.

10. $x^2 = 24 + 4x$.

15. $x^2 + \frac{7}{3}x = 2$.

16. $x^2 - \frac{3}{2}x = \frac{27}{4}$.

196. **The Hindu method of completing the square.** In case the coefficient of x^2 is not unity, as in $3x^2 + 8x = 4$, both members may be divided by this coefficient, and the solution is then like that of Exs. 1 and 2 above.

However, the following method is sometimes desirable:

$$3x^2 + 8x = 4. \quad (1)$$

Multiplying each member of (1) by $4 \cdot 3 = 12$,

$$36x^2 + 96x = 48. \quad (2)$$

Completing the square,

$$36x^2 + 96x + 8^2 = 48 + 64 = 112. \quad (3)$$

$$\text{Taking square roots, } 6x + 8 = \pm \sqrt{112} = \pm 4\sqrt{7}. \quad (4)$$

$$\text{Hence } x = -\frac{4}{3} \pm \frac{4}{3}\sqrt{7}. \quad (5)$$

The advantage of this form of solution is that fractions are avoided until the last step, and *the number added to complete the square is the square of the coefficient of x in the original equation.*

This is called the **Hindu method** of completing the square.

Note. Fractions would also be avoided in the above solution if equation (1) were multiplied by 3 instead of $4 \cdot 3$. This is the case only when the coefficient of x is an *even* number.

EXERCISES

In the solution of the following equations the roots which contain surds may be left in simplified radical form.

1. $2x^2 + 3x = 2.$
2. $3x^2 + 5x = 2.$
3. $3x = 9 - 2x^2.$
4. $6x + 1 = -3x^2.$
5. $2x^2 = 5x + 3.$
6. $4x = 2x^2 - 1.$
7. $2x^2 - 3x = 14.$
8. $3x^2 = 9 + 2x.$
9. $4x^2 = 2x + 1.$
10. $6x - 1 = 3x^2.$
11. $2x^2 + 4x = 23.$
12. $3x^2 - 7 = 4x.$
13. $2x^2 - 5 = 3x.$
14. $4x^2 = 6x - 1.$
15. $2x = 1 - 5x^2.$
16. $3x - 20 = -2x^2.$
17. $2x + 3x^2 = 9.$
18. $4x^2 - 1 = 3x.$
19. $4x = 7 - 2x^2.$
20. $2x + 1 = 5x^2.$
21. $3x^2 + 4x = 7.$
22. $3x + 9 = 2x^2.$
23. $2x - 1 = -4x^2.$
24. $5x^2 + 16x = -2.$

CHECKING RESULTS IN QUADRATIC EQUATIONS

197. **Illustrative Examples.** 1. Solving $x^2 - 7x + 12 = 0$, we get $x = 4$ and $x = 3$.

What is the *sum* of these roots? How does this compare with the coefficient of x ? What is the *product* of these roots? How does this compare with the known term of the equation?

2. Solving $x^2 - 6x + 4 = 0$, we get $x = 3 + \sqrt{5}$ and $x = 3 - \sqrt{5}$.

What is the *sum* of these roots? How does this sum compare with the coefficient of x ? What is the product of these roots? How does this *product* compare with the known term of the equation?

Solve the following equations. In each case compare the product of the roots with the known term and the sum of the roots with the coefficient of x .

1. $x^2 - 5x + 3 = 0$.

4. $x^2 - 4x - 8 = 0$.

2. $x^2 + 3x + 2 = 0$.

5. $x^2 + 6x - 3 = 0$.

3. $x^2 + 9x + 8 = 0$.

6. $x^2 - 8x = 6$.

198. These exercises are illustrations of a general rule for all quadratics written in the form $x^2 + px + q = 0$, in which the coefficient of the squared term is $+1$, and all terms are transposed to the left member; namely:

The sum of the roots is equal to the coefficient of x with its sign changed and the product of the roots is equal to the known term.

This may be used to check the results obtained in solving a quadratic. Note in particular that *before applying the test* you must put the equation into the specified form.

Note that in the radical form the product of the roots is *exactly* the known term, but when the roots are approximated in the decimal form, then their product is only *approximately* equal to the known term.

EXERCISES

Solve the following equations and check all results by means of § 198.

1. $4x^2 + 1 = 8x.$

13. $3x^2 + 2x = 5.$

2. $2x^2 - 3x = 20.$

14. $2 + 3x = 2x^2.$

3. $2x^2 - 3 = -5x.$

15. $8x + 1 = -4x^2.$

4. $3x^2 + 4x = 8.$

16. $8 + 4x = 3x^2.$

5. $10 - 4x = 5x^2.$

17. $10 + 4x = 5x^2.$

6. $1 + 4x^2 = -6x.$

18. $2 + 5x = 3x^2.$

7. $5 - 3x = 2x^2.$

19. $3x + 14 = 2x^2.$

8. $7 + 4x = 2x^2.$

20. $3x^2 - 2x = 5.$

9. $6x^2 + 12x = 2.$

21. $2x^2 + 4x = 1.$

10. $6x^2 - 12x = -2.$

22. $4x^2 + 3x = 1.$

11. $6x^2 + 12x = -2.$

23. $2x^2 - 4x = 23.$

12. $6x^2 - 12x = 2.$

24. $2x^2 - 3x = 1.$

SOLUTION OF THE QUADRATIC BY FORMULA

199. Solve the equation

$$ax^2 + bx + c = 0. \quad (1)$$

By $S, M,$

$$4a^2x^2 + 4abx = -4ac. \quad (2)$$

Completing the square, $4a^2x^2 + 4abx + b^2 = b^2 - 4ac.$

(3)

Taking square roots,

$$2ax + b = \pm \sqrt{b^2 - 4ac}. \quad (4)$$

By $S, D,$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (5)$$

Calling the two values of x in the result x_1 and x_2 we have,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Verify these results by means of § 198.

Any quadratic equation may be reduced to the form of (1) by simplifying and collecting the coefficients of x^2 and x . Hence any quadratic equation may be solved by substituting in the formulas just obtained.

EXAMPLE. Solve $2x^2 - 4x + 1 = 0$.

In this case $a = 2, b = -4, c = 1$.

$$\text{Hence } x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2}$$

$$\text{From which } x_1 = \frac{2 + \sqrt{2}}{2}, \quad x_2 = \frac{2 - \sqrt{2}}{2}.$$

Check the results by § 198.

200. A quadratic equation may be proposed for solution which has no roots expressible in terms of the numbers of arithmetic or algebra thus far studied.

EXAMPLE. Solve $x^2 + 4x = -8$. (1)

By A, $x^2 + 4x + 4 = -4$. (2)

Taking square roots, $x + 2 = \pm \sqrt{-4}$. (3)

$\sqrt{-4}$ is unknown to us as a number symbol, since there is no number thus far considered whose square equals -4 . (See Principle IX.) Such symbols are defined and used in the Advanced Course, and are called **imaginary numbers**. Any quadratic equation which gives rise to such a solution is to be interpreted as stating some condition not satisfied by any number so far studied.

EXERCISES

Leave the surds or imaginaries in simplified radical form.

- | | |
|----------------------------------|-------------------------------------|
| 1. $7 - 3x = 5x^2$. | 7. $x^2 - 8 + 3x = -15x$. |
| 2. $51x - 33 = 3x^2$. | 8. $11x^2 - 49x + 57 = 0$. |
| 3. $14x + 8 - x^2 = 52 - 3x^2$. | 9. $3x^2 + 18 - 16x = 5$. |
| 4. $12 - 51x = 36 + 6x^2$. | 10. $37 - 4x^2 - 12x = 79 - 5x^2$. |
| 5. $5x + x^2 + 8 = 0$. | 11. $10x^2 + 41 + 7x = 44$. |
| 6. $5x^2 - 31x = -6$. | 12. $45 + 3x^2 - 85 - 2x = 0$. |

MISCELLANEOUS QUADRATICS

Solve as many as possible of the following equations by factoring. When this is not convenient, use the formula of § 199, or complete the square independently in each case. It is important to check the results in the radical form as in § 198.

1. $x^2 + 11x = 210$.
2. $5x^2 - 3x = 4$.
3. $7x + 3x^2 - 18 = 0$
4. $2 = 5x + 7x^2$.
5. $6x - 11x^2 = -7$.
6. $-51 + 42x - 3x^2 = 0$.
7. $3x^2 + 3x = 2x + 4$.
8. $13 - 8x + 3x^2 = 0$.
9. $2x^2 + 11x = 32x - x^2 - 27$.
10. $176 + 3x - x^2 = 2x$.
11. $x^2 + 6x - 54 = 0$.
12. $5x^2 + 9x + 12 = 4x^2 + x$.
13. $2x^2 - 4x - 25 = 0$.
14. $7x^2 + 11x = 6$.
15. $2x^2 - 11x + 5 = 0$.
16. $2x^2 - 11x = 6$.
17. $25x - 95 = x^2$.
18. $11x^2 - 42x = 2$.
19. $x^2 - 8x - 4 = x - 22$.
20. $8x^2 + 5x = -8$.
21. $2x^2 + 3x - 3 = 12x + 2$.
22. $3x^2 - 7x = 10$.
23. $17x + 31 + 2x^2 = 0$.
24. $18 - 41x = 3 + x^2$.
25. $10x + 25 = 5 - 2x - x^2$.
26. $3x - 59 + x^2 = 0$.
27. $5x^2 + 7x - 6 = 0$.
28. $x^2 + 12 = 7x$.
29. $8x - 5x^2 = 2$.
30. $5x + 3x^2 - 22 = 0$.
31. $50 + 20x + x^2 = 5x$.
32. $x^2 + x + 4 = 0$.
33. $20x + 2x^2 + 42 = 33x + x^2$.
34. $17x - 3x^2 = -6$.
35. $8x + 5x^2 = -2$.
36. $10 + 15x + x^2 = 26x$.
37. $3x^2 - 2x - 7 = 0$.
38. $5x^2 - 9x - 18 = 0$.
39. $7x - 7x^2 + 24 = 0$.
40. $31 + 2x + x^2 = 0$.
41. $7x^2 + 7x - 5x^2 + 20 = x^2 - 2x + 2$.
42. $5x^2 + 3x - 7 = (x-1)(x+2)$.
43. $(x-3)^2 - (2x-1)(2x+1) + 7 = 0$.
44. $3x + (3x-2)^2 = 4x^2 - 1$.
45. $9x^2 - (2x-1)^2 = (x+3)^2$.
46. $7x^2 = 5x - (x-2)^2 + 7$.
47. $(3x-2)(3x+2) = (2x-8)^2$.
48. $5x - 9x^2 + 8(x-x^2) = 4$.

DRILL EXERCISES

1. Divide $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ by $x + y$.
2. Divide $x^6 - y^6$ by $x^2 + xy + y^2$.
3. Divide $x^5 - y^5$ by $x^3 + x^2y + xy^2 + y^3$.
4. Divide $x^9 + x^6y^3 + y^9$ by $x^3 + y^3$.

Find the square roots of:

5. $x^6 + 2x^4 + 2x^3 + x^2 + 2x + 1$.
6. $x^5 + 2x^4 + 3x^3 + 2x^2 + 1$.
7. $x^6 + 4x^5 + 10x^4 + 16x^3 + 17x^2 + 12x + 4$.
8. $4a^4 - 12a^3 - 7a^2 + 24a + 16$.

Rationalize the denominators of the following:

9. $\frac{3}{\sqrt{7} - \sqrt{4}}$.
11. $\frac{a+b}{\sqrt{a} + \sqrt{b}}$.
13. $\frac{\sqrt{8}+7}{\sqrt{8}-7}$.
10. $\frac{\sqrt{7} + \sqrt{2}}{\sqrt{7} - \sqrt{2}}$.
12. $\frac{2 + \sqrt{3}}{4 - \sqrt{3}}$.
14. $\frac{\frac{1}{2}\sqrt{2} - 1}{\frac{1}{3}\sqrt{2} + 1}$.
15. $\begin{cases} x + ay = c, \\ bx + y = d. \end{cases}$
17. $\begin{cases} ax + by = 1, \\ cx + dy = 4. \end{cases}$
16. $\begin{cases} ax + 3y = 2c, \\ bx - 2y = 3d. \end{cases}$
18. $\begin{cases} ax - by = c, \\ cx + dy = e. \end{cases}$
19. $\begin{cases} \frac{1}{x} + \frac{1}{y} = a, \\ \frac{1}{y} + \frac{1}{z} = b, \\ \frac{1}{z} + \frac{1}{x} = c. \end{cases}$
20. $\begin{cases} x + y + z = a, \\ 2x - 2y + 2z = b, \\ 3x - y - z = c. \end{cases}$
21. $\begin{cases} ax - y + bz = a, \\ x + ay - z = 1, \\ bx - y + az = b. \end{cases}$

22. If w and l are the length and width of a rectangle, express in symbols the length of its diagonal.

23. The lengths of the two sides of a right triangle are 16 and 24 respectively. Express the length of the hypotenuse in the simplest form without approximating a square root.

SYSTEMS INVOLVING QUADRATICS

201. The solution of two equations in two variables, one of which is linear and the other quadratic, can be reduced to the solution of a quadratic equation in one variable.

EXAMPLE. Solve $\begin{cases} x + y = 3, \\ 3x^2 - y^2 = 14. \end{cases}$ (1)

(2)

From (1), $y = 3 - x$. (3)

Substituting in (2) and reducing,

$$2x^2 + 6x - 23 = 0. \quad (4)$$

Substituting in the formula § 199,

$$x = \frac{-6 \pm \sqrt{36 - 4 \cdot 2(-23)}}{4} = \frac{-3 \pm \sqrt{55}}{2} \quad (5)$$

Hence

$$x_1 = 2.21 \text{ and } x_2 = -5.21.$$

Substituting these values of x in (1) we have as the approximate roots,

$$\left. \begin{array}{l} x_1 = 2.21 \\ y_1 = 0.79 \end{array} \right\} \text{ and } \left. \begin{array}{l} x_2 = -5.21 \\ y_2 = 8.21 \end{array} \right\}.$$

y_1 and y_2 are here used to designate the values of y , which correspond to x_1 and x_2 respectively.

EXERCISES

In the above manner solve the following systems of equations, finding in each case two pairs of roots. In the case of roots which are surds, find the approximate results to two places of decimals.

1. $\begin{cases} x - y = 1. \\ x^2 + y^2 = 13. \end{cases}$

5. $\begin{cases} x + 4y = 26. \\ x^2 - y^2 = 11. \end{cases}$

2. $\begin{cases} x + y = 9. \\ x^2 + y^2 = 41. \end{cases}$

Ans. $x_1 = 6, y_1 = 5.$
 $x_2 = -9\frac{7}{8}, y_2 = 8\frac{1}{8}.$

3. $\begin{cases} x + y = 13. \\ xy = 42. \end{cases}$

6. $\begin{cases} \frac{x^2}{9} - \frac{y^2}{4} = 3. \\ x - y = 4. \end{cases}$

4. $\begin{cases} 3x - y = 5. \\ x^2 + y^2 = 25. \end{cases}$

$$7. \begin{cases} x - y = 1. \\ \frac{x^2}{36} + \frac{y^2}{16} = \frac{1}{4}. \end{cases}$$

$$8. \begin{cases} 2x + y = 5. \\ 3x^2 - 5y^2 = 7. \end{cases}$$

$$9. \begin{cases} x - y = 3. \\ x^2 - 3y^2 = 13. \end{cases}$$

$$10. \begin{cases} x - 3y = 1. \\ y^2 + 2x^2 = 33. \end{cases}$$

$$\text{Ans. } x_1 = 4, y_1 = 1. \\ x_2 = -3\frac{1}{3}, y_2 = -1\frac{2}{3}.$$

$$11. \begin{cases} 3x - 4y = 1. \\ x^2 - y^2 = 24. \end{cases}$$

$$12. \begin{cases} x + y = 4. \\ 2x^2 - 3xy + y^2 = 8. \end{cases}$$

$$13. \begin{cases} x - y = 1. \\ 4x^2 + 2xy - y^2 = 19. \end{cases}$$

$$14. \begin{cases} 5x + y = 12. \\ 2x^2 - 3xy + y^2 = 0. \end{cases}$$

$$15. \begin{cases} x - 2y = 3. \\ 2y^2 - x^2 = 4. \end{cases}$$

$$\text{Ans. } x_1 = -6.16, y_1 = -4.58 \\ x_2 = .16, y_2 = -1.42.$$

$$16. \begin{cases} x + y = 9. \\ x^2 - 2y^2 = -7. \end{cases}$$

$$17. \begin{cases} y - 2x = 5. \\ y^2 - 3xy = 16. \end{cases}$$

$$\text{Ans. } x_1 = 3.71, y_1 = 12.42. \\ x_2 = -1.21, y_2 = 2.58.$$

$$18. \begin{cases} 2y - 3x = 0. \\ y^2 + x^2 = 52. \end{cases}$$

$$19. \begin{cases} y - 2x = 5. \\ x^2 + y^2 = 40. \end{cases}$$

$$20. \begin{cases} x - 4y = 12. \\ 3x^2 + 2xy - 6y = 44. \end{cases}$$

$$\text{Ans. } x_1 = 4, y_1 = -2. \\ x_2 = -1\frac{2}{3}, y_2 = -3\frac{1}{3}.$$

PROBLEMS

In each problem find the two roots of the quadratic equation, and determine whether both are applicable to the problem:

1. The area of a window is 2016 square inches and the perimeter of the frame is 180 inches. Find the dimensions of the window.

2. The area of a rectangular city block, including the sidewalk, is 19,200 square yards. The length of the sidewalk around the block when measured on the side next the street is 560 yards. Find the dimensions of the block.

3. A farmer starts to plow around a rectangular field which contains 48 acres. The length of the first furrow around the field is 376 rods. Find the dimensions of the field.

4. A rectangular blackboard contains 38 square feet and its perimeter is 27 feet. Find the dimensions of the board.

5. A park is 120 rods long and 80 rods wide. It is decided to double the area of the park, still keeping it rectangular, by adding strips of equal width to one end and one side. Find the width of the strips.

6. A fancy quilt is 72 inches long and 56 inches wide. It is decided to increase its area 10 square feet by adding a border. Find the width of the border.

7. A city block is 400 by 480 feet when measured to the outer edge of the sidewalk. At 4 cents per square foot it costs \$416.64 to lay a sidewalk around the block. Find the width of the walk.

8. A farmer starts cutting grain around a field 120 rods long and 70 rods wide. How wide a strip must be cut to make 12 acres?

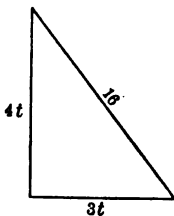
9. The sides of a right triangle are 6 and 8 inches respectively. How much must be added to each side so as to increase the hypotenuse 10 inches, it being understood that each side is increased by the same amount?

10. A rectangular lot is 16 by 12 rods. How wide a strip must be added to one end and one side to obtain a rectangular lot whose diagonal is 1 rod greater?

11. A picture is 15 inches by 20 inches. How wide a frame must be added to increase the diagonal 3 inches?

12. An athletic field is 800 feet long and 600 feet wide. The field is to be extended by the same amount in length and width so that the longest possible straight course (the diagonal) shall be increased by 100 feet. How much is the field extended in each direction? *Ans.* 71.36 feet.

13. *A* and *B* start from a certain cross roads at the same time, *A* going north 4 miles per hour and *B* going east 3 miles per hour. In how many hours will they be 16 miles apart, measuring in a straight line across country?



Let t equal the required number of hours.

Then $(4t)^2 + (3t)^2 = 16^2 = 256.$

$$16t^2 + 9t^2 = 256.$$

$$25t^2 = 256.$$

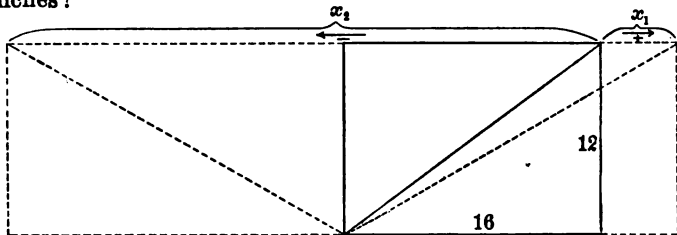
$$5t = \pm 16.$$

$$t = \pm 3\frac{1}{5}.$$

The solution $t = -3\frac{1}{5}$ may be interpreted as meaning that if the two men were traveling along these roads in the same direction before reaching the cross road, they would be 16 miles apart $3\frac{1}{5}$ hours before meeting.

14. In the preceding problem if *A* goes 5 miles per hour and *B* 4 miles per hour, in how many hours will they be 24 miles apart? *Ans.* 3.75.

15. A rectangle is 12 inches wide and 16 inches long. How much must be added to the length to increase the diagonal 4 inches?



Let x = number of inches to be added to the length. The diagonal of the original rectangle is $\sqrt{12^2 + 16^2} = 20$. Hence the diagonal of the required rectangle is 24.

Then

$$12^2 + (16 + x)^2 = 24^2,$$

or

$$x^2 + 32x - 176 = 0.$$

Solving,

$$x_1 = -16 + 12\sqrt{3} = 4.78,$$

and

$$x_2 = -16 - 12\sqrt{3} = -36.78.$$

The negative solution obtained here may be taken to mean that if the rectangle is extended in the *opposite* direction from the fixed corner, we shall get a rectangle which has the required diagonal. See the figure.

16. How much must the *width* of the rectangle in problem 15 be extended so as to increase the diagonal by 4?

17. A trunk 30 inches long is just large enough to permit an umbrella 36 inches long to lie diagonally on the bottom. How much must the length of the trunk be increased if it is to accommodate a gun 4 inches longer than the umbrella?

18. A rectangle is 21 inches long and 20 inches wide. The length of the rectangle is decreased twice as much as the width, thereby decreasing the length of the diagonal 4 inches. Find the dimensions of the new rectangle.

19. In a rectangular table cover 24 by 30 inches there are two strips of drawn work of equal width running at right angles through the center of the piece. What is the width of these strips if the drawn work covers one tenth of the whole piece?

20. A certain university campus is 100 rods long and 80 rods wide. There are two driveways running through the center of the campus at right angles to each other and parallel to the sides. What is the width of these driveways if their combined area is 356 square rods?

21. A farm is 320 rods long and 280 rods wide. There is a road 2 rods wide running around the boundary of the farm and lying entirely within it. There is also a road 2 rods wide running across the farm parallel to the ends. What is the area of the farm exclusive of the roads?

22. A rectangular park is 480 rods long and 360 rods wide. A walk is laid out completely around the park, and a drive through the length of the park parallel to the sides. What is the width of the walk if the drive is 3 times as wide as the walk and the combined area of the walk and the drive is 3110 square rods?

23. The sum of the sides of a right triangle is 18 and the length of the hypotenuse is 16. Find the length of each side.

24. The length of a fence around a rectangular athletic field is 1400 feet, and the longest straight track possible on the field is 500 feet. Find the dimensions of the field.

Using 100 feet for the unit of measure, the equations are

$$\begin{cases} x + y = 7, \\ x^2 + y^2 = 25. \end{cases}$$

25. The difference between the sides of a right triangle is 8 and the hypotenuse is 42. Find the lengths of the sides.

26. A room is 5 feet longer than it is wide, and the distance between two opposite corners is 25 feet. Find the length and width of the room.

27. One side of a right triangle is 8 feet, and the hypotenuse is 2 feet more than twice the other side. Find the length of the hypotenuse and of the remaining side.

28. A vacant corner lot has a 50-foot frontage on one street. What is the frontage on the other street if the distance between opposite corners along the diagonal is 110 feet less than twice this frontage.

29. The sum of the squares of two consecutive integers is 13,945. Find the numbers.

30. The product of two consecutive integers is 4422. Find the numbers.

31. A square piece of tin is made into an open box, containing 864 cubic inches, by cutting out a 6-inch square from each corner of the tin and then turning up the sides. Find the dimensions of the original piece of tin.

32. A rectangular piece of tin is 8 inches longer than it is wide. By cutting out a 7-inch square from each corner and turning up the sides, an open box containing 1260 cubic inches is formed. Find the dimensions of the original piece of tin.

33. By cutting out a square 8 inches on a side from each corner of a sheet of metal and turning up the sides, we obtain an open box such that the area of the sides and ends is 4 times the area of the bottom. Find the dimensions of the original sheet if it is twice as long as it is wide. *Ans.* 41.48 in. by 20.74 in.

34. An open box whose bottom is a square has a lateral area which is 400 square inches more than the area of the bottom. Find the other dimensions of the box if it is 10 inches high. (By lateral area is meant the sum of the areas of the four sides.)

35. A box whose bottom is 4 times as long as it is wide has a lateral area 600 square inches less than 4 times the area of the bottom. Find the dimensions of the bottom if the box is 6 inches high.

REVIEW QUESTIONS

1. Explain the method of solving a quadratic equation by factoring. Can you apply this method to solve the equation $(x+1)(x-2)=5$? Explain.

2. Explain the method of solving a quadratic equation by completing the square. What is the Hindu method and what are its advantages?

3. How many roots has a quadratic equation? Find the roots of $x^2 + 2x + 5 = 0$. Are these roots in the form of any numbers thus far studied? What are such roots called? Solve $x^2 + 2x - 5 = 0$. What are these roots called?

4. How are the roots of the equation $x^2 + px + q = 0$ related to p and q ? How may this be used in checking the solutions?

5. Do both roots of a quadratic equation necessarily satisfy the conditions of the *problem* from which such an equation may be derived? In checking the solution of a problem is it sufficient to make the test alone in the equation derived from the problem?

DRILL EXERCISES

Simplify each of the following as much as possible without approximating roots:

1. $\sqrt{32} + \sqrt{72} - \sqrt{50}.$

2. $2a\sqrt{a^3b} - b\sqrt{b^3a} + \sqrt{ab}.$

3. $\sqrt{a^3 + 2a^2b + ab^2} + \sqrt{a^3 - 2a^2b + ab^2} - 2\sqrt{a^3}.$

4. $\sqrt{a^3 - a^2b} - \sqrt{b^3a - b^3} + \sqrt{a^3b^3 - a^2b^3}.$

5. $\sqrt{(x+y)(x^2-y^2)} - \sqrt{(x+y)^2(x-y)} - \sqrt{x^3-x^2y}.$

Solve the following equations:

6. $\sqrt{x^2-8} + x = 8.$

8. $\sqrt{2a-1} = 7 - \sqrt{2a+6}.$

7. $\sqrt{5a-24} + 4 = \sqrt{5a}.$

9. $\sqrt{x+2} = \sqrt{x-6} + 2\sqrt{x-5}.$

Rationalize the denominators of the following:

10. $\frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}}.$

12. $\frac{a + \sqrt{b}}{a - \sqrt{b}}.$

11. $\frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}}.$

13. $\frac{\sqrt{x-a} + b}{\sqrt{x+a} - c}.$

Solve:

14.
$$\begin{cases} 3ax - by = 2, \\ 2x + 3by = 6. \end{cases}$$

15.
$$\begin{cases} 2x - 3by = c, \\ 2ax - 5y = d. \end{cases}$$

16.
$$\begin{cases} x - y - 3z = -6, \\ 2x + y - z = 11, \\ -x + 3y + z = 16. \end{cases}$$

17.
$$\begin{cases} 2x - y + 3z = 20, \\ x + 4y - z = -2, \\ 5x + y - 6z = 6. \end{cases}$$

18. Divide $x^5 - 3x^4 - 18x^3 + 24x^2 + 52x - 21$ by $x^2 + x - 7$.

19. Divide $6a^5 + 5a^4 - 60a^3 + 4a^2 + 71a + 28$ by $3a^2 - 5a - 4$.

Find the square roots of:

20. $16x^8 - 40x^6y^3 + x^4y^6 + 30x^2y^9 + 9y^{12}.$

21. $4m^6 - 20m^5 + 9m^4 + 52m^3 - 14m^2 - 24m + 9.$

CHAPTER XIII

ALGEBRAIC FRACTIONS

COMMON FACTORS

Simple algebraic fractions have already been studied in Chapter V, where they were treated exactly as fractions in arithmetic. This was sufficient for all our purposes up to this point. We now take up a more formal study of fractions.

202. If a number is a factor of each of two or more numbers, it is said to be a **common factor** of these numbers.

Thus, 8 is a common factor of 16 and 48, and 12 is a common factor of 12, 36, and 48.

If each of a given set of numbers is separated into prime factors, any common factor which they may have is at once apparent.

Illustrative Example. Find the common factors of $10(x+y)^2(x-y)$, $5(x+y)(x^2-y^2)$, and $15(x+y)(x^2-y^2)$.

Factoring, $10(x+y)^2(x-y) = 2 \cdot 5(x+y)(x+y)(x-y)$.

$$5(x+y)(x^2-y^2) = 5(x+y)(x+y)(x-y).$$

$$15(x+y)(x^2-y^2) = 3 \cdot 5(x+y)(x-y)(x+xy+y^2).$$

The common *prime* factors are 5, $x+y$, and $x-y$. The other common factors, obtained by combining these, are $5(x+y)$, $5(x-y)$ and $5(x+y)(x-y)$. The last factor, $5(x+y)(x-y)$ is called the highest common factor.

The name *highest* instead of *greatest* is used in algebra referring to the number of prime factors which enter into it. Thus, x^2 is of higher degree than x , although if $x = \frac{1}{2}$, x^2 is not greater than x .

203. Definition. The product of all the common prime factors is called the **highest common factor**. This is usually abbreviated to H. C. F.

EXERCISES

Find the H. C. F. of the following sets of expressions :

1. $x - y, x^2 - y^2, x^2 - 2xy + y^2$.
2. $x^2 + 2x + 1, 3x + 6x^2 + 3x^3$.
3. $x^2 + 4x + 4, x^3 - 6x - 16$.
4. $x^2 - 8x + 16, x^2 + 10x - 56$.
5. $a^3 - b^3, a^2 - 2ab + b^2$.
6. $x^3 + y^3, x^2 - y^2, x^2 + 2xy + y^2$.
7. $x^2 - 7x + 12, ax - 3a - bx + 3b$.
8. $a^2 - 13a + 42, a^3 - 216, a^2 - a - 30$.
9. $27 + y^3, y^3 + 9y + 18, y^3 - 9$.
10. $b^2 + 7b - 30, b^2 + 11b - 42, b^2 - b - 6$.
11. $a^3 + 2a^2 + a, a^2 + a, a^3 + 5a^2 + 4a$.
12. $x^3 + y^3, x^3 + x^2y + xy^2 + y^3$.
13. $x^4 + 3x^3 + 2x^2, x^3 + x^2, x^4 + 7x^3 + 6x^2$.
14. $x^2 - 11x + 30, xz - 5z + x^2 - 5x$.
15. $m^3 - n^3, 2x^2m^2 + 2x^2mn + 2x^2n^2$.
16. $x^2 - 1, x^3 - 1, x^3 - 13x + 12$.
17. $1 - 64x^3, 1 - 16x^2, 5 - 2z - 20x + 8xz$.
18. $1 + 125a^3, 1 + 10a + 25a^2, 1 - 25a^2$.
19. $ac - ax + 3bc - 3bx, a^3 + 27b^3$.
20. $5c - 2, 5ac + 20c - 2a - 8$.
21. $4x^4 - x^2, 2x^4 + x^3 - x^2, 2x^4 - 3x^3 + x^2$.
22. $3a^3 - 3a, 3a^3 - 6a^2 + 3a, 6a^3 + 12a^2 - 15a$.
23. $6x - 10xy + 4xy^2, 18x - 8xy^2, 54x - 16xy^3$.
24. $3x^5 + 9x^4 - 3x^3, 5x^2y^2 + 15xy^2 - 5y^2, 7ax^2 + 21ax - 7a$.
25. $18x^3 - 57x^2 + 30x, 9x^2 - 15x^2 + 6x, 18x^3 - 39x^2 + 18x$.

COMMON MULTIPLES

204. A number is said to be a **multiple of any of its factors**. In particular any number is a multiple of itself and of one.

Thus, 18 is a multiple of 1, 2, 3, 6, 9, and 18, but not of 12. $3a^2x^2$ is a multiple of 3, $3x$, $3x^2$, etc.

Since a multiple of a number is divisible by that number, it must contain as a factor every factor of that number.

E.g., 108 is a multiple of 54 and contains as factors all the factors of 54, namely 3, 3, 3, and 2, and also 2, 6, 9, 18, and 54.

Definition. A number is a **common multiple** of two or more numbers if it is a multiple of each of them.

Thus, 18 is a common multiple of 6, 9, and 18. Evidently $3 \cdot 18$, $4 \cdot 18$, $5 \cdot 18$, etc. are also common multiples of 6, 9, and 18. Of all these common multiples 18 is called the *lowest* common multiple.

205. The process of finding the lowest common multiple of a set of expressions is shown as follows:

Illustrative Example. Find the lowest common multiple of

$$x^2 - y^2; \quad x^2 + 2xy + y^2; \quad \text{and} \quad x^2 - 2xy + y^2.$$

$$\text{Factoring,} \quad x^2 - y^2 = (x - y)(x + y). \quad (1)$$

$$x^2 + 2xy + y^2 = (x + y)(x + y). \quad (2)$$

$$x^2 - 2xy + y^2 = (x - y)(x - y). \quad (3)$$

In order that an expression may be a multiple of (1) it must contain the factors $x - y$ and $x + y$. To be a common multiple of (1) and (2) it must contain an additional factor $x + y$, that is, it must contain $(x - y)$, $(x + y)$, $(x + y)$. To be a common multiple of (1), (2), and (3) it must contain an additional factor $x - y$, that is, it must contain $(x - y)$, $(x + y)$, $(x + y)$, $(x - y)$. The product $(x - y)(x + y)(x + y)(x - y) = (x - y)^2(x + y)^2$ is called the lowest common multiple of (1), (2), and (3), since it is the common multiple which contains the smallest number of prime factors.

In general, the process may be described as follows: to obtain the lowest common multiple of a set of expressions, factor each expression into prime factors; use all factors of the first expression together with those factors of the second which are not in the first, those of the third which are not in the first and second, etc.

It is evident that in this manner we obtain a product which is a common multiple of the given expressions, but such that if any one of these factors is omitted, it will cease to be a multiple of some one of the expressions; that is, it will no longer be a common multiple of them *all*.

Thus, if in the example above either of the factors $x - y$ is omitted, the product will no longer be a multiple of $x^2 - 2xy + y^2$.

206. Definition. The lowest common multiple of a set of expressions is that common multiple which contains the smallest number of prime factors. The lowest common multiple is usually abbreviated to L. C. M.

EXERCISES

Find the L. C. M. of the following expressions:

1. $2 \cdot 3 \cdot 4$, $3 \cdot 7 \cdot 8$; $2^3 \cdot 3 \cdot 4$.
2. $5x^2y^4$, $10x^3y$, $25x^2y$.
3. $2ab$, $6a^2$, $4b^2c$.
4. $x^2 - y^2$, $x^2 - 2xy + y^2$.
5. $x - y$, $x + y$, $x^2 - y^2$.
6. $4 - x^2$, $2 - x$, $2 + x$.
7. $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$.
8. $x^3 + 3x + 2$, $x^2 - 4$, $x^2 - 1$.
9. $25x^3 - 1$, $125x^3 - 1$.
10. $2x^2 - 7x + 6$, $4x^2 - 11x + 6$.
11. $x^3 - y^3$, $x - y$, $x^2 + xy + y^2$.
12. $x^3 - y^3$, $x^3 + y^3$, $x^2 - y^2$.

13. $5x^2 + 7x - 6, x^2 - 15x - 34.$

14. $x^3 + y^3, x^3 - y^3, (x - y)^2.$

15. $3abc, a^2 - 4ac + 4c^2, a - 2c.$

16. $x^2 - 1, x + 1, x^2 + 8x + 7$

17. $4x^2y - 44x^2y + 120xy, 3a^3x^3 - 22a^3x + 35a^3.$

18. $x^2 + 2xy + y^2, 2ax^2 - 10ax + 12a.$

19. $3bx^2 - 21bx + 36b, x^3 - 5x + 4.$

20. $5a^2b^2 - 5a^2c^2, b^2 + 2bc + b + c + c^2.$

21. $15c^2ax^2 + 16c^2ax + c^2a, 2cax^2 + 10cax + 8ca.$

REDUCTION OF FRACTIONS TO LOWEST TERMS

207. By Principle XV any factor common to the numerator and denominator of a fraction may be cancelled. That is,

$$\frac{ak}{bk} = \frac{a}{b}.$$

Thus, $\frac{2 \cdot 3 \cdot 4 \cdot 5}{3 \cdot 7 \cdot 11} = \frac{2 \cdot 4 \cdot 5}{7 \cdot 11}; \frac{2^3 \cdot 3^2 \cdot 4x}{2^4 \cdot 3 \cdot 4^2} = \frac{3x}{2 \cdot 4} = \frac{3x}{8};$

$$\frac{x^2 - 7x + 12}{x^2 - 5x + 6} = \frac{(x-3)(x-4)}{(x-2)(x-3)} = \frac{x-4}{x-2}.$$

If the terms of a fraction have no common factor, the fraction is said to be in its lowest terms.

EXERCISES

Reduce the following fractions to lowest terms:

1. $\frac{3 \cdot 9^2 \cdot 2^6}{2^4 \cdot 5^3 \cdot 9^4}.$ 4. $\frac{a^4b^3}{a^2b^2}.$ 7. $\frac{x^3 - y^3}{2x^2 - 3xy + y^2}.$

2. $\frac{4a^4b^{12}c^3}{8a^3b^4c^4}.$ 5. $\frac{x^2 + 2xy + y^2}{x^2 - y^2}.$ 8. $\frac{64 - b^3}{16 - 8b + b^2}.$

3. $\frac{x^2y^3z^4}{xy^2z^3}.$ 6. $\frac{x^2 + 7x - 30}{x^2 - 7x + 12}.$ 9. $\frac{x^3 + 27z^3}{xy - 5x + 3yz - 15z}.$

2. 10. $\frac{1 - 216c^3}{x - 4y - 6cx + 24cy}$. 19. $\frac{4x^4 - 28x^3 + 48x^2}{2x^4 - 8x^3 + 6x^2}$.
3. 11. $\frac{14bz - 2bx + ax - 7az}{x^2 - 49z^2}$. 20. $\frac{9a^2b^4 + 18a^2b^3c + 9a^2b^2c^2}{3ab^3 - 3abc^2}$.
4. 12. $\frac{3a^2 - 29a + 56}{63 - 9a - 7m + ma}$. 21. $\frac{7xy^2 - 133xy + 126x}{15xy^2 - 36xy + 21x}$.
13. $\frac{a(x-y)^3}{(x^2-y^2)(x-y)}$. 22. $\frac{20x^3 + 20x^2y + 5xy^2}{60x^3 - 15x^2y^2}$.
14. $\frac{x^3 + 27}{4x^2 + 24x + 36}$. 23. $\frac{3ab^4 - 3ab^2c^2}{27a^3b^2 + 27a^3bc}$.
15. $\frac{a^3 - 3a - 3b + ab}{(a^2 - b^2)(a - 3)}$. 24. $\frac{4a^5 - 42a^2 + 20a}{2a^4b^6 - 20a^3b^6}$.
16. $\frac{2^7 \cdot 3^3 \cdot 5^4 - 2^8 \cdot 3^4 \cdot 5^7}{2^4 \cdot 3^2 \cdot 5^5 - 2^5 \cdot 3^3 \cdot 5^2}$. 25. $\frac{(x-1)(x-2)(x-3)(x-4)}{(x-1)(x-3)(x-3)(x-4)}$.
17. $\frac{5x^3y^4 - 12x^2y^5 + 7x^2y^3}{6x^5y^2 + 3x^2y^2}$. 26. $\frac{(x^2 - y^2)(x^2 + 2xy + y^2)}{(x^2 - 2xy + y^2)(x + y)}$.
18. $\frac{5c + 10b - bc - 2b^2}{8c^3 + 64b^3}$. 27. $\frac{(x^2 - 1)(x^2 + 1)(3x^2 + 3)}{3(x^4 - 1)}$.

REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR

208. By the formula $\frac{a}{b} = \frac{ak}{bk}$ any factor may be introduced into the numerator and denominator of a fraction.

In this manner any fraction may be changed into an equal fraction whose denominator is any given multiple of the denominator of the given fraction.

$$E.g. \quad \frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5}, \quad \frac{a-b}{a+b} = \frac{(a-b)(a+b)}{(a+b)(a+b)} = \frac{a^2 - b^2}{(a+b)^2}.$$

Any two or more fractions may therefore be changed into respectively equal fractions which shall have a *common denominator*, namely, a common multiple of the denominators of the given fractions.

Illustrative Example. Reduce $\frac{x-1}{x+1}$, $\frac{x+1}{x-1}$, $\frac{2x+3}{x^2-1}$ to fractions having a common denominator.

The L. C. M. of the denominators is $(x-1)(x+1)$. Multiply the numerator and denominator of each fraction by an expression which will make the denominator of each new fraction $(x-1)(x+1)$.

$$\begin{aligned}\text{Thus, } \frac{x-1}{x+1} &= \frac{(x-1)(x-1)}{(x+1)(x-1)} = \frac{x^2-2x+1}{(x+1)(x-1)}; \\ \frac{x+1}{x-1} &= \frac{(x+1)(x+1)}{(x-1)(x+1)} = \frac{(x+1)^2}{(x+1)(x-1)}; \\ \frac{2x+3}{x^2-1} &= \frac{2x+3}{(x+1)(x-1)}.\end{aligned}$$

It is best to *indicate* the multiplication in the common denominator, since this makes it more easily apparent by what expression the numerator and denominator of a fraction must be multiplied in order to reduce it to a fraction with the required denominator.

209. The Three Signs of a Fraction. It should be noticed that there are three signs in connection with a fraction: the sign of the fraction itself, the sign of the numerator, and the sign of the denominator. Any two of these signs may be changed simultaneously without changing the value of the fraction.

$$\text{Thus, (1) } \frac{a}{b} = \frac{-a}{-b}. \quad (2) \frac{a}{b} = -\frac{a}{-b}. \quad (3) \frac{a}{b} = -\frac{-a}{b}.$$

Show why each of these statements is true. What principles are involved in each case?

How may the sign of the numerator of a fraction be changed if it is in the form of a polynomial? If it is in the form of a product of several factors? Take care to distinguish these two cases.

This is useful in cases like the following:

Reduce $\frac{x+1}{1-x}$, $\frac{x}{x^2-1}$, and $\frac{1}{x+1}$ to fractions having a common denominator.

$$\frac{x+1}{1-x} = \frac{-x-1}{x-1} = \frac{(x+1)(-x-1)}{(x+1)(x-1)} = \frac{-x^2-2x-1}{x^2-1},$$

$$\frac{x}{x^2-1} = \frac{x}{x^2-1}, \text{ and } \frac{1}{x+1} = \frac{x-1}{(x+1)(x-1)} = \frac{x-1}{x^2-1}.$$

Apply the sign changes shown in (1), (2), and (3) above to each of the following fractions:

$$(1) \frac{a}{b-c}, \quad (2) \frac{a-b}{c-d}, \quad (3) \frac{-a}{c+d}, \quad (4) \frac{(a-b)(c-d)}{-x},$$

$$(5) \frac{1}{(1-x)(x+2)}, \quad (6) -\frac{b-a}{d-c}, \quad (7) -\frac{-c}{(d-a)(b-a)}.$$

EXERCISES

Reduce each of the following sets of fractions to equivalent fractions having a common denominator.

$$1. \frac{x+3}{x-y}, \frac{4}{x^2-2xy+y^2}. \quad 4. \frac{2}{3-x}, \frac{x-1}{x+1}, \frac{x+1}{x-3}.$$

$$2. \frac{a-1}{a^2-b^2}, \frac{a+1}{a^2+2ab+b^2}. \quad 5. \frac{a+b}{b-a}, \frac{a-b}{(a+b)^2}, \frac{a}{a^2-b^2}.$$

$$3. \frac{3-x}{x^2-9x+20}, \frac{x+4}{7x^2-26x-8}. \quad 6. \frac{1}{x^2-3x-4}, \frac{1}{x^2+3x+2}.$$

$$7. \frac{a+1}{a^2-2ab+b^2}, \frac{a-1}{a^2+2ab+b^2}. \quad 8. \frac{1}{a^3-b^3}, \frac{1}{b-a}, \frac{1}{a^2+ab+b^2}.$$

$$9. \frac{a}{5a^2-4a-12}, \frac{b}{a^2+4a-12}, \frac{c}{a-2}.$$

$$10. \frac{x-2}{x^2-5x-6}, \frac{x+2}{x^2+12x-108}, \frac{x-1}{x^2+19x+18}.$$

$$11. \frac{mr}{m-1}, \frac{d}{1+n}. \quad 12. \frac{Rr}{(R+r)(m-1)}, \frac{1}{R+r}. \quad 13. \frac{a}{n-a}, \frac{b}{n-b}.$$

$$14. \frac{W(T-Q)}{w(Q-t)}, \frac{V}{w}.$$

$$16. \frac{RA}{a-A}, \frac{1}{R+r}, \frac{1}{A-a}.$$

$$15. \frac{V}{V-v}, \frac{V}{V+v}, \frac{1}{V^2-v^2}.$$

$$17. \frac{R}{x}, \frac{r}{y}, \frac{1}{x-y}, \frac{Rx}{x+y}.$$

210. Since any number may be written as a fraction with the denominator 1, the above process may be used to reduce an integral expression to the form of a fraction having any desired denominator.

$$\text{Thus,} \quad 3 = \frac{3 \cdot 5}{5}; \quad x - y = \frac{(x-y)(x^2-1)}{x^2-1}, \text{ etc.}$$

It is sometimes convenient to reduce expressions, some of which are not fractions, to the form of fractions having a common denominator.

Illustrative Example. Reduce $5x$, $\frac{5x-1}{x^2-1}$, $\frac{2x-y}{x-1}$, to fractions having a common denominator. The lowest common denominator is x^2-1 .

$$\text{Thus,} \quad 5x = \frac{5x(x^2-1)}{x^2-1} = \frac{5x^3-5x}{x^2-1}, \quad \frac{5x-1}{x^2-1} = \frac{5x-1}{x^2-1},$$

$$\frac{2x-y}{x-1} = \frac{(2x-y)(x+1)}{(x-1)(x+1)} = \frac{2x^2+2x-yx-y}{x^2-1}.$$

EXERCISES

Reduce the following expressions to fractions having a common denominator:

$$1. \quad 5, x-y, \frac{5x-3}{x^2+2xy+y^2}.$$

$$4. \quad x^2+xy+y^2, \frac{x+y}{x-y}.$$

$$2. \quad \frac{3a-c}{x-y}, \frac{2b-c}{x+y}, 2c+2.$$

$$5. \quad x^2-xy+y^2, x^2-y^2, \frac{1}{x+y}.$$

$$3. \quad 1+a+a^2, \frac{a+1}{a-1}.$$

$$6. \ x^4 + x^2y^2 + y^4, \frac{x}{x-y}, \frac{y}{x+y}.$$

$$7. \ 3a - 2b - c, \frac{5}{a-b}, \frac{2}{b-c}.$$

$$8. \ x^4 - 1, \ x^2 - 1, \frac{x+1}{x-1}.$$

$$9. \ x^2 + 2xy + y^2, \frac{1}{x+y}, \frac{1}{1-x}.$$

$$10. \ x+y, \ x-y, \frac{x-y}{x^2+y^2}, \frac{x+1}{x-y}.$$

$$11. \ \frac{RA}{a-A}, r.$$

$$12. \ \frac{RA}{a-A}, R-r.$$

$$13. \ T, \frac{Q+t}{2}.$$

$$14. \ \frac{W(q-t)}{w}, sT, \frac{s(t-q)}{2}.$$

$$15. \ H, \frac{hd}{D}.$$

ADDITION AND SUBTRACTION OF FRACTIONS

211. By Principle III read in the reverse order:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \text{ and } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

If fractions which are to be added or subtracted do not have a common denominator, they must be reduced to this form.

EXAMPLE. Add $\frac{a-b}{a+b}$ and $\frac{a^2+2ab+b^2}{a^2-2ab+b^2}$.

Reducing the fractions to the common denominator

$$(a-b)(a-b)(a+b),$$

we have

$$\frac{a-b}{a+b} = \frac{(a-b)(a-b)(a-b)}{(a+b)(a-b)(a-b)} = \frac{a^3 - 3a^2b + 3ab^2 - b^3}{(a+b)(a-b)(a-b)},$$

$$\text{and } \frac{a^2+2ab+b^2}{a^2-2ab+b^2} = \frac{(a+b)(a^2+2ab+b^2)}{(a+b)(a-b)(a-b)} = \frac{a^3 + 3a^2b + 3ab^2 + b^3}{(a+b)(a-b)(a-b)}.$$

Adding the numerators, we have $2a^3 + 6ab^2$; whence the sum of the fractions is

$$\frac{2a^3 + 6ab^2}{(a+b)(a-b)(a-b)}.$$

EXERCISES

Perform the following additions and subtractions:

$$1. \frac{3}{4} + \frac{2}{7} + \frac{a+b}{3a-b}.$$

$$2. \frac{x-y}{(x+y)^2} - \frac{x-y}{x^2-y^2}.$$

$$3. \frac{x^3-9x+18}{x^2-13x+36} + \frac{x}{4-x}.$$

$$4. \frac{2}{3} + \frac{a}{a+b} + \frac{b}{a-b}.$$

$$5. \frac{3}{2^3 \cdot 3^2} + \frac{5}{2^2 \cdot 3^4} - \frac{2}{2^4 \cdot 3^3}.$$

$$6. \frac{a^2-9b^2}{a^2+6ab+9b^2} - \frac{a^2-6ab}{a^2-9b^2}.$$

$$7. \frac{x+y}{x-y} + \frac{x-y}{x+y} - 2.$$

$$8. \frac{2x^2}{x^2-y^2} - \frac{y}{x-y} - \frac{y}{x+y}.$$

$$9. \frac{a+1}{a^2+a+1} - \frac{a-1}{a^2-a+1}.$$

$$10. \frac{b}{1-b} + \frac{b}{1+b^2} - \frac{b^2}{1-b^2}.$$

$$11. \frac{x+1}{x-2} + \frac{x+1}{x+2} + \frac{3x+2}{x^2-4}.$$

$$12. \frac{x-1}{x+1} - \frac{x+1}{x-1} + \frac{x^2-5}{x^2-1}.$$

$$13. \frac{y^2}{y^2-1} + \frac{y}{y+1} - \frac{y}{1-y}.$$

$$14. \frac{1}{x} - \frac{1}{y} - \frac{1}{x-y} + \frac{-1}{x+y}.$$

$$15. \frac{1}{2} - \frac{b}{b-a} + \frac{2a^2}{b^2-a^2} + \frac{-a}{b+a}.$$

$$16. \frac{x^2 + 4xy}{x^2 + y^2} + \frac{1}{x+y} - \frac{x}{x^2 - xy + y^2}.$$

$$17. \frac{1}{1-x^3} + \frac{1}{1-x} - \frac{1}{1+x+x^2}.$$

$$18. \frac{a-3}{a^2-3a+2} - \frac{a-1}{a^2-5a+6} + \frac{a}{a^2-4a+3}.$$

$$19. \frac{x}{x^2-5x-14} + \frac{2}{x-7} - \frac{x}{x^2-9x+14}.$$

$$20. \frac{a}{ac+ad-bc-bd} - \frac{b}{a^2-2ab+b^2}.$$

$$21. \frac{2}{a-5} + \frac{39}{a^2+3a-40} + \frac{3}{a+8}.$$

$$22. \frac{1}{y^2+8y+16} - \frac{1}{y(y+4)} + \frac{4}{y^2(y+4)}.$$

$$23. \frac{x}{x^2+4x-60} + \frac{x}{x^2-4x-12}.$$

$$24. \frac{a-c}{a^2-c^2} + \frac{c-a}{a^2+2ac+c^2} - \frac{2}{a-c}.$$

$$25. \frac{9}{x^2+7x-18} - \frac{8}{x^2+6x-16}.$$

$$26. \frac{a+2}{a^2-a-6} + \frac{a-4}{a^2-7a+12} - \frac{a+2}{a^2-2a-8}.$$

$$27. \frac{2}{x^2-11x+30} - \frac{1}{x^2-36} + \frac{1}{x^2-25}.$$

$$28. \frac{1}{(x-1)(x+2)} + \frac{1}{(x+2)(x-3)} + \frac{1}{(x-1)(3-x)}.$$

$$29. \frac{1}{(a-b)(b-c)} - \frac{1}{(b-a)(c-d)} + \frac{1}{(b-c)(c-d)}.$$

$$30. \frac{4}{a-3} - \frac{a-1}{a^2+3a+9} + \frac{a^2-38a-3}{a^3-27}.$$

DRILL EXERCISES

Find the square root of each of the following:

1. $25a^2 + c^2 + 9b^2 - 10ac + 30ab - 6bc$.
2. $x^2 + y^2 + 4z^2 + 4v^2 - 2xy + 4xz - 4xv - 4yz + 4yv - 8zv$.
3. $x^6 - 2x^5 - x^4 + 3x^3 + 2x + 1$.
4. $x^4 - 6x^3 + 13x^2 - 12x + 4$.
5. Divide $6x^4 + x^3 + 12x^2 + 8$ by $2x^2 - x + 2$.
6. Divide $3x^6 + 4x^5 - x^4 + 6x^3 - 12x^2 + 8x - 12$ by $3x^2 - 2x + 3$.
7. Divide $3a^7 - 5a^5 + 8a^3 + 2a^2 - 18a + 12$ by $a^3 - a + 2$.
8. Divide $6a^4 + 10a^3 - 9a^2 + 11a - 6$ by $2a^2 + 4a - 3$.

To each of the following binomials add one squared term so as to make it a trinomial square:

- | | | |
|--------------------|--------------------|--------------------|
| 9. $4a^2 + 8a$. | 12. $25x^2 - 7x$. | 15. $3x^2 - 4x$. |
| 10. $9a^2 + 30a$. | 13. $7b^2 - 3b$. | 16. $7x^2 - 11x$. |
| 11. $x^2 + 3x$. | 14. $16b^2 - 7b$. | 17. $8x^2 + 7x$. |

Reduce the solution of each of the following to simplest form without approximating any roots:

- | | |
|---------------------------|----------------------------------|
| 18. $7x^2 = 27$. | 24. $7ax^2 = 98$. |
| 19. $5x^2 = 108$. | 25. $2(a+b)x^2 = 300$. |
| 20. $2x^2 = 3$. | 26. $x^2 = \frac{162}{125ab}$. |
| 21. $3a^2 = 343$. | 27. $x^2 = \frac{128}{72cd^2}$. |
| 22. $5x^2 = 8a^2b$. | 28. $(a-b)x^2 = a+b$. |
| 23. $2x^2 = 27a(a+b)^2$. | |

Solve each of the following and test results by the method given in § 198.

- | | |
|----------------------------|------------------------------|
| 29. $x^2 - 7x + 9 = 0$. | 33. $x^2 + 2bx = 3c$. |
| 30. $2x^2 - 5x + 2 = 0$. | 34. $2x^2 - 5ax = a^2$. |
| 31. $7x^2 + 18x - 3 = 0$. | 35. $ax^2 + 2bx = 3c$. |
| 32. $x^2 - 12x + 16 = 0$. | 36. $4a^2x^2 + 6bx = 3b^2$. |

MULTIPLICATION AND DIVISION OF FRACTIONS

212. Since a fraction is an indicated quotient, and since multiplying the dividend or dividing the divisor multiplies the quotient, it follows that the *product of a fraction and an integral expression is obtained by multiplying the numerator or dividing the denominator by the integral expression.*

$$\text{Thus, } 4 \cdot \frac{7}{8} = \frac{4 \cdot 7}{8} = \frac{7}{2} \text{ or } 4 \cdot \frac{7}{8} = \frac{7}{8 \div 4} = \frac{7}{2}.$$

$$\text{That is, in general, } a \cdot \frac{b}{c} = \frac{a \cdot b}{c} = \frac{ab}{c} = \frac{a}{c \div b}.$$

It is best to factor completely the expressions to be multiplied and to keep them in the factored form until all possible cancellations have been made.

EXERCISES

Find the following indicated products and reduce the fractions to the simplest form:

$$1. (1-a) \times \frac{1+a+a^2}{a-1}.$$

$$7. (1-x^2) \times \frac{1-x}{1+x+x^2}.$$

$$2. (x^2-y^2) \times \frac{x+y}{x-y}.$$

$$8. (27a^3-1) \times \frac{a+1}{9a^2+3a+1}.$$

$$3. (x^2-2xa+a^2) \times \frac{x+a}{x-a}.$$

$$9. (a^2+ab+b^2) \times \frac{a-b}{a^2-b^2}.$$

$$4. \frac{3x-1}{x^2-5x+6} \times (x^2-11x+18).$$

$$10. (1-a+a^2) \times \frac{a+1}{a^2+1}.$$

$$5. \frac{a^2-4a-3}{a^2-8a+16} \times (a^2-5a+4).$$

$$11. \frac{x+1}{x^4-1} \times \frac{x^2-1}{(x+1)^2}.$$

$$6. (x^2+9x+18) \times \frac{x-5}{x^2-2x-15}.$$

$$12. \frac{a+b}{a^6-b^6} \times \frac{a^3+b^3}{a^2-b^2}.$$

213. Since multiplying the divisor or dividing the dividend divides the quotient, it follows that *a fraction is divided by an integral expression by dividing the numerator or multiplying the denominator by the integral expression.*

$$\text{Thus, } \frac{8}{9} \div 2 = \frac{8 \div 2}{9} = \frac{4}{9} \quad \text{or} \quad \frac{8}{9} \div 2 = \frac{8}{9} \times \frac{1}{2} = \frac{8}{9 \times 2} = \frac{4}{9}.$$

$$\text{That is, in general, } \frac{a}{b} \div c = \frac{a}{b \cdot c} = \frac{a \div c}{b}.$$

EXERCISES

Find the following indicated quotients and reduce the fractions to their lowest terms:

1. $\frac{x^3 - y^3}{x + y} \div (x^2 + xy + y^2).$
2. $\frac{x^3 + y^3}{x - y} \div (x^2 - y^2).$
3. $\frac{x^3 + 4x + 4}{x^2 - 2x + 1} \div (x^2 - 4).$
4. $\frac{1 - 27x^3}{1 - 9x^2} \div (1 + 3x + 9x^2).$
5. $\frac{x^3 + 2x - 35}{x^2 + 10x + 21} \div (x^2 - 4x - 5).$
6. $\frac{x^3 - 16x + 39}{x^2 - 8x + 15} \div (x^2 - x - 156).$
7. $\frac{ac + bc - ad - bd}{xy - 4x - 3y + 12} \div (cx - 3c - dx + 3d).$
8. $\frac{x^2 + ax + bx + ab}{x^2 + ax - 3x - 3b} \div (x^2 + ax - 5x - 5a).$
9. $\frac{mr + ms - nr - ns}{mx - m - nx + n} \div (3r - xr + 3s - xs).$
10. $\frac{x^3 - 3x - 88}{x^2 - 9x - 22} \div (x^2 + 9x + 2).$

TO MULTIPLY A FRACTION BY A FRACTION

214. To multiply a number by the quotient of two numbers is the same as to multiply by the dividend and then divide the product by the divisor.

$$\text{Thus,} \quad \frac{4}{9} \cdot \frac{3}{2} = \left(\frac{4}{9} \cdot 3 \right) \div 2 = \frac{4}{3} \div 2 = \frac{2}{3}.$$

$$\text{In general,} \quad \frac{a}{b} \cdot \frac{c}{d} = \left(\frac{a}{b} \cdot c \right) \div d = \frac{ac}{b} \div d = \frac{ac}{bd}.$$

That is, the *product of two algebraic fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

Illustrative Example. Multiply $\frac{x^2-1}{x^2-7x+10}$ by $\frac{x^2-3x+2}{x^2+2x+1}$ and reduce the resulting fraction to its lowest terms.

$$\begin{aligned} \frac{x^2-1}{x^2-7x+10} \times \frac{x^2-3x+2}{x^2+2x+1} &= \frac{(x-1)(\cancel{x+1})(x-2)(x-1)}{(x-2)(x-5)(\cancel{x+1})(x+1)} \\ &= \frac{(x-1)(x-1)}{(x-5)(x+1)} = \frac{x^2-2x+1}{x^2-4x-5}. \end{aligned}$$

It is desirable to resolve each numerator and denominator into prime factors, and then cancel all common factors before performing any multiplication.

EXERCISES

Find the following indicated products and reduce each fraction to its lowest terms:

$$1. \quad \frac{3x^2y^2}{2yz^2} \times \frac{6az}{9x^2}.$$

$$3. \quad \frac{12c^2b}{5(c^2-b^2)} \times \frac{35(c^2+cb+b^2)}{14c^2b^2}.$$

$$2. \quad \frac{5a(a-b)}{3c(a+b)} \times \frac{9(a+b)^2}{15(a^2-b^2)}.$$

$$4. \quad \frac{y^2+3y+2}{y^2-5y+6} \times \frac{y^2-7y+12}{y^2+8y+7}.$$

5. $\frac{3^2 \cdot 4^3}{5^2 \cdot 2^4} \times \frac{10 \cdot 2}{3^4} \quad \times 7. \frac{x^2 - x}{x^2 - 1} \times \frac{2x^2 + 4x + 2}{3x^2 + 6x}.$
6. $\frac{3^2 \cdot 2^3}{5^2} \times \frac{5^4 \cdot 7^2}{3^4 \cdot 2^3} \times \frac{6 \cdot 3^2}{5^4 \cdot 7^3} \quad \times 8. \frac{a^2 - 10a + 16}{a^2 + 6a + 9} \times \frac{a + 3}{a^2 - 4}.$
9. $\frac{(x+y)^2 - z^2}{x^2 + xy - xz} \times \frac{x}{(x-z)^2 - y^2} \times \frac{(x-y)^2 - z^2}{xy - y^2 - yz}.$
10. $\frac{3(x+4)^2}{4(x+4)(x-7)} \times \frac{(x-7)^2}{3(x+4)(x-7)}.$
11. $\frac{a^3 + 5a^2 - 36a}{a^2 - 7a - 144} \times \frac{(a-16)(a-3)}{a(a-4)(a+2)}.$
12. $\frac{3a(a+7)(a-5)}{7b(a+3)(a+7)} \times \frac{b(a+3)(a+10)}{a(a-5)(a-10)}.$
13. $\frac{3t^2 - 2t - 1}{2t^2 + t - 1} \times \frac{2t^2 + 5t - 3}{3t^2 + 7t + 2} \times \frac{4t^2 + 10t + 4}{4t^2 - 2t - 2}.$
14. $\frac{6x^2 - 7x + 2}{10x^2 - 7x + 1} \times \frac{6x^2 - 5x - 1}{6x^2 + x - 1} \times \frac{10x^2 + 3x - 1}{5x^2 - 4x - 1}.$
15. $\frac{4b^2 - 17b + 4}{6b^2 - 7b + 2} \times \frac{10b^2 - 21b + 9}{5b^2 - 23b + 12} \times \frac{3b^2 - 5b + 2}{4b^2 - 5b + 1}.$

TO DIVIDE A FRACTION BY A FRACTION

215. To divide a number by the quotient of two numbers is the same as to divide by the dividend and then multiply the result by the divisor.

$$\text{Thus,} \quad \frac{4}{9} \div \frac{2}{3} = \left(\frac{4}{9} \div 2 \right) \times 3 = \frac{2}{9} \times 3 = \frac{2}{3}$$

$$\text{In general,} \quad \frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \div c \right) \times d = \left(\frac{a}{b \cdot c} \right) \times d = \frac{ad}{bc}.$$

That is, a number is divided by a fraction by inverting the fraction and multiplying by the new fraction thus obtained.

EXERCISES

Perform the following indicated divisions, and reduce the resulting fractions to their lowest terms:

$$1. \frac{a^3 + b^3}{a^3 - 9b^2} \div \frac{a + b}{a + 3b} \quad \text{3} \quad \frac{x^2 - 6x - 16}{x^2 + 4x - 21} \div \frac{x^2 + 9x + 14}{x^2 - 8x + 15}$$

$$2. \frac{x^2 + x - 2}{x^2 - 3x} \div \frac{x^3 + 2x^2}{x^2 + 9x - 36} \quad 4. \quad \frac{x^2 - 1}{x^2 - 4x - 5} \div \frac{x^2 + 2x - 3}{x^2 - 25}$$

$$5. \frac{x^2 + 9xy + 18y^2}{x^2 - 9xy + 20y^2} \div \frac{x^2 + 6xy + 9y^2}{xy^2 - 4y^3}$$

$$6. \frac{3a^4 - 9a^3 - 54a^2}{9a^3 - 117a^2 + 378a} \div \frac{a^3 + 8a^2 + 15a}{3a^2 - 33a + 84}$$

$$7. \frac{a^2 - 11a + 30}{a^3 - 6a^2 + 9a} \times \frac{a^2 - 3a}{a^2 - 25} \div \frac{a^2 - 9}{a^2 + 2a - 15}$$

$$8. \frac{x^3 - 10x + 21}{x^2 + x - 56} \div \frac{x^3 - 8x + 15}{x^2 + 4x - 32}$$

$$9. \frac{a^2 - b^2}{ab^2x} \times \frac{b(a-b)}{a^2 + 2ab + b^2} \div \frac{b(a+b)}{a^2 - 2ab + b^2}$$

$$10. \frac{a+b}{ab} \times \frac{a^2 - b^2}{3(a^2 + b^2)} \div \frac{(a-b)^2(a+b)^2}{3a^2y + 3ay^2}$$

$$11. \frac{5x^4 - 5x^3}{7x^2 - 56x - 63} \div \frac{x^4 - 9x^2 + 8x^3}{14x^2 + 14x - 1260}$$

$$12. \frac{8y^2(y+4)(y+5)}{2^4(y+5)(y-7)} \div \frac{y(y+4)(y+8)}{2^3(y-7)(y+11)}$$

$$13. \frac{x^3(x-2)^2}{(x+2)^2} \times \frac{(x+2)(x-3)}{(x-2)(x-7)} \div \frac{x^2(x-3)(x-5)}{(x-5)(x-7)}$$

$$14. \frac{a^2b^2(c+5)(c-4)}{(c-4)(c-8)} \times \frac{(c-8)(c+9)}{(c+4)(c+7)} \div \frac{ab(c+9)(c-1)}{(c+7)(c+1)}$$

$$15. \frac{21x^3 + 23x - 20}{10x^2 - 27x + 5} \times \frac{6x^3 - 11x - 10}{3x^2 + 2x - 5} \div \frac{7x^2 + 17x - 12}{5x^2 + 9x - 2}$$

COMPLEX FRACTIONS

216. Sometimes fractions occur whose numerators or denominators, or both, contain fractions.

$$E.g. \quad \frac{1 + \frac{1}{a}}{1 - \frac{1}{a}} \text{ and } \frac{\frac{1}{x+1} + \frac{1}{x-1}}{\frac{1}{x-1} - \frac{1}{x+1}}.$$

Such fractions are called **complex fractions**. A complex fraction is said to be **simplified** when it is reduced to an equal fraction whose numerator and denominator are in the integral form.

$$Ex. 1. \quad \frac{1 + \frac{1}{a}}{1 - \frac{1}{a}} = \frac{\frac{a+1}{a}}{\frac{a-1}{a}} = \frac{a+1}{a} \div \frac{a-1}{a} = \frac{a+1}{a-1}.$$

This result may also be obtained directly by multiplying both terms of the given fraction by a , finding at once $\frac{a+1}{a-1}$.

$$Ex. 2. \quad \frac{\frac{1}{x+1} + \frac{1}{x-1}}{\frac{1}{x-1} - \frac{1}{x+1}} = \frac{\frac{x-1+x+1}{x^2-1}}{\frac{x+1-x+1}{x^2-1}} = \frac{2x}{x^2-1} \times \frac{x^2-1}{2} = x.$$

By multiplying the terms of the given fraction by $(x+1)(x-1)$ we may also get directly $\frac{x-1+x+1}{x+1-x+1} = x$.

Make a rule for finding the expression by which numerator and denominator of a complex fraction may be multiplied so as to reduce it directly to a simple fraction. Apply this rule to each of the following.

$$Ex. 1. \quad \frac{\frac{1}{2} + \frac{1}{3} - \frac{1}{4}}{\frac{2}{3} + \frac{5}{6} - \frac{1}{2}}$$

$$Ex. 2. \quad \frac{\frac{1}{4} + \frac{1}{8} + \frac{1}{2}}{\frac{3}{2} - \frac{1}{4} - \frac{1}{8}}$$

EXERCISES

Reduce each of the following complex fractions to its simplest form:

$$1. \frac{1+x}{1+\frac{1}{x}}$$

$$4. \frac{\frac{m+n+1}{3}}{\frac{m-n-1}{2}}$$

$$7. \frac{\frac{a^3-8b^3}{27}}{3a-2b}$$

$$2. \frac{1-\frac{a}{b}}{1+a}$$

$$5. \frac{4+\frac{a+b}{2}}{4-\frac{a-b}{2}}$$

$$8. \frac{\frac{M}{D^3}}{1+\frac{a^3}{D^3}}$$

$$3. \frac{x+\frac{x}{2}}{x-\frac{x}{2}}$$

$$6. \frac{\frac{x^2-y^2}{4}}{\frac{x+y}{2}}$$

$$9. \frac{\frac{H}{c}-\frac{hd}{cD}}{\frac{1+t}{c}}$$

$$10. \frac{\frac{x}{x-3}-\frac{x}{x+3}}{\frac{x+2}{x-3}-\frac{x-2}{x+3}}$$

$$13. \frac{a}{a+\frac{a}{a+x}}$$

$$11. \frac{\frac{1}{x-4}+\frac{1}{x-3}}{\frac{x}{x^2-7x+12}}$$

$$14. \frac{\frac{x-3}{x+4}-\frac{x-4}{x+2}}{x-4}$$

$$12. \frac{1}{1+\frac{1}{1+x}}$$

$$15. \frac{\frac{2}{x-1}-\frac{1}{x-2}}{\frac{3}{x-3}-\frac{4}{x-1}}$$

$$16. \frac{\frac{2}{x-3}+\frac{2x^2+2}{x-2x^3}}{\frac{1}{x-3}-\frac{1}{1-2x^3}}$$

DRILL EXERCISES

Solve the following systems of equations :

$$1. \begin{cases} x^2 + y^2 = 25, \\ x - y = 4. \end{cases}$$

$$4. \begin{cases} xy - y^2 = 14, \\ x + y = 4. \end{cases}$$

$$2. \begin{cases} 2x^2 - 3xy = 12, \\ x + 2y = 4. \end{cases}$$

$$5. \begin{cases} x^2 - xy = 8, \\ x + y = -2. \end{cases}$$

$$3. \begin{cases} x^2 - y^2 = 8, \\ x - 2y = 3. \end{cases}$$

$$6. \begin{cases} 2x^2 - 3y^2 = 7, \\ x - y = 5. \end{cases}$$

$$7. \begin{cases} 3x - 7by = 2, \\ 2ax + 2y = 4. \end{cases}$$

$$9. \begin{cases} \frac{7x-3}{2} - \frac{ay+1}{3} = 1, \\ \frac{bx+5}{4} + \frac{y-3}{3} = 6. \end{cases}$$

$$8. \begin{cases} 6x + 3y = 1, \\ 5ax - 2by = c. \end{cases}$$

$$10. \begin{cases} \frac{2by}{3a} - 3x = 2, \\ 5y + \frac{2ax}{b} = 5. \end{cases}$$

$$11. \begin{cases} 2x - y - z = 8, \\ 3x + 2y + z = 24, \\ -x - 3y + 5z = 16. \end{cases}$$

$$12. \begin{cases} 8x - 3y - z = 0, \\ 2x + 2y + 3z = 10, \\ -x + y + 6z = 8. \end{cases}$$

13. A picture inside the frame is 12 inches long and 8 inches wide. If the frame is a inches wide, express its area in terms of a .

14. If h is the length of the hypotenuse of a right triangle and a the length of one side, express the length of the third side in terms of h and a .

15. A rectangular piece of tin is w inches wide and l inches long. If a square a inches on a side is cut out of each corner, express in terms of w , l , and a the volume of a box formed by turning up the sides.

16. A farmer plows a strip a rods wide around a rectangular field w rods wide and l rods long. Express in terms of w , l , and a the area plowed.

CHAPTER XIV

RATIO, VARIATION, AND PROPORTION

217. Definitions. A fraction is often called a **ratio**. Thus $\frac{a}{b}$ may be read *the ratio of a to b*, and is also written $a : b$.

The numerator is called the **antecedent** of the ratio, and the denominator the **consequent**. The antecedent and consequent are called the **terms** of the ratio.

An equation, each of whose members is a ratio, is called a **proportion**.

Thus, $\frac{a}{b} = \frac{c}{d}$ is a proportion, and is also written $a : b = c : d$.

It is read *the ratio of a to b equals the ratio of c to d*, or briefly, *a is to b as c is to d*.

The four numbers a , b , c , and d are said to be in **proportion**, a and d are called the **extremes** of the proportion, and b and c the **means**.

218. If in the graph of $y = 2x$, or $\frac{y}{x} = 2$, we think of a point as moving along the line, we see that its coördinates x and y vary, but always so that their ratio is constant, namely 2.

Thus if x_1 and x_2 are any two values of x , and y_1 and y_2 are the corresponding values of y , given by the equation $y = 2x$, we have $\frac{y_1}{x_1} = 2$ and $\frac{y_2}{x_2} = 2$, and hence the proportion $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.

219. Definition. When any two variables are connected in such a way that their ratio is constant, either one is said to **vary directly as the other**. This is usually written in the form

$$y = kx,$$

where k is a constant and it is read **y varies directly as x**.

In this case the variables x and y are said to be *proportional*. We also say that y is proportional to x or x is proportional to y .

Ex. 1. The formula for a moving body $s = vt$ states that if the velocity is constant, the distance varies as the time. Hence if t_1 and t_2 are two values of t , and s_1 and s_2 the corresponding values of s , we have the proportion

$$\frac{s_1}{t_1} = \frac{s_2}{t_2}.$$

In this case the **space** is said to be proportional to the **time**.

Ex. 2. From the formula for percentage, $p = br$ form a proportion, if r is fixed and p and b vary; also if b is fixed and p and r vary.

Here the **percentage** is proportional to the **base** if the rate is constant, and proportional to the **rate** if the base is constant.

Ex. 3. From the formula for interest, $i = prt$ form a proportion if p and r are fixed and i and t vary; if p and t are fixed and i and r vary; if r and t are fixed and i and p vary.

To what is i proportional if p and r are fixed? if r and t are fixed? if p and t are fixed?

EXERCISES

1. If y varies as x and $y = 3$ when $x = 7$, find y when $x = 12$.

Solution. We have the proportion $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ in which are given $x_1 = 7$, $y_1 = 3$, $x_2 = 12$, to find y_2 .

Substituting, we have $\frac{3}{7} = \frac{y_2}{12}$ or $y_2 = 5\frac{1}{4}$.

2. If s varies as t and if $s = 40$ when $t = 5$, find s when $t = 16$.

3. If percentage varies as the base and $p = 50$ when $b = 1000$, find b when $p = 175$.

IMPORTANT PROPERTIES OF A PROPORTION

220. The following examples show some of the properties of a proportion.

1. If, in the proportion $\frac{a}{b} = \frac{c}{d}$, both members of the equation be multiplied by bd , we have $ad = bc$.

That is: *If four numbers are in proportion, the product of the means equals the product of the extremes.*

2. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

Hint. Divide $1 = 1$ by the members of the given equation.

This process is called taking the proportion by **inversion**.

3. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Hint. Multiply both members of the given equation by $\frac{b}{c}$.

To find this multiplier we inquire by what expression $\frac{a}{b}$ must be multiplied to give $\frac{a}{c}$.

This process is called taking the proportion by **alternation**.

4. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

Hint. Add 1 to both members of the given equation.

This process is called taking the proportion by **addition**.

5. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

Hint. Subtract 1 from each member of the given equation.

This process is called taking the proportion by **subtraction**.

6. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

Hint. Divide the members of the equation obtained under 4 by the members of the one obtained under 5.

This process is called taking a proportion by **addition** and **subtraction**.

7. Show that if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then $\frac{a+c+e}{b+d+f} = \frac{a}{b}$.

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$; then $a = bk$, $c = dk$, $e = fk$.

Hence, $a + c + e = bk + dk + fk = (b + d + f)k$,

and $\frac{a+c+e}{b+d+f} = k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

That is, *If several ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

EXERCISES

1. If $ad = bc$, show that $\frac{a}{b} = \frac{c}{d}$. *Hint.* Divide by bd .

2. If $ad = bc$, show that $\frac{a}{c} = \frac{b}{d}$. 3. If $ad = bc$, show that $\frac{d}{c} = \frac{b}{a}$.

4. If $ad = bc$, show that $\frac{d}{b} = \frac{c}{a}$.

5. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{a} = \frac{c+d}{c}$.

6. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{a} = \frac{c-d}{c}$.

7. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{a+b} = \frac{c-d}{c+d}$.

8. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{a-b}{c-d}$.

9. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{a}{c}$.

10. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+c}{b+d} = \frac{a}{b}$.

11. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{c-d} = \frac{a}{c}$.

12. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-c}{b-d} = \frac{a}{b}$.

13. Solve the equation $\frac{a}{b} = \frac{c}{d}$ for each letter in terms of all the others. If $a=3$, $b=5$, $c=8$, find d . If $b=7$, $c=9$, $d=3$, find a . If $c=13$, $d=2$, $a=5$, find b . If $d=50$, $a=3$, $b=-7$, find c .

14. If $\frac{a}{b} = \frac{c}{x}$, then x is said to be a **fourth proportional** to a , b , and c .

Find a fourth proportional to 3, 5, and 7; also to 9, 5, and 1, and to 3, -2, and -5.

15. If $\frac{a}{x} = \frac{x}{b}$, then x is called a **mean proportional** between a and b .

Solve the equation $\frac{a}{x} = \frac{x}{b}$ for x in terms of a and b . Show that there are two solutions, each of which is a mean proportional between a and b .

Find two mean proportionals between 4 and 9; also between 5 and 125, and between -4 and -36.

16. Which is the greater ratio, $\frac{5+3d}{5+4d}$ or $\frac{5+4d}{5+5d}$?

Hint. Reduce the fractions to a common denominator and compare numerators. (d is a positive number.)

17. Which is the greater ratio, $\frac{a+7b}{a+8b}$ or $\frac{a+9b}{a+10b}$?

18. Which is the greater ratio, $\frac{a}{b}$ or $\frac{a+c}{b+c}$, if b and c are positive, and a less than b ? a equal to b ? a greater than b ?

19. Find two numbers in the ratio of 3 to 5 whose sum is 160.

20. Find two numbers in the ratio of 2 to 7 whose sum is -108.

21. Find two numbers in the ratio of 3 to -4 whose sum is -15.

22. What number added to each of the terms of the ratio $\frac{4}{7}$ makes it equal to $\frac{5}{6}$?

23. What number must be added to each term of the ratio $\frac{1}{11}$ to make it equal to the ratio $\frac{2}{5}$?

24. What number added to each of the numbers 3, 5, 7, 10, will make the sums in proportion, when taken in the given order?

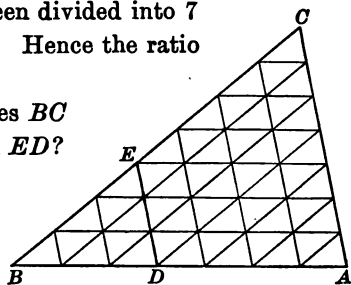
25. Two numbers are in the ratio of 2 to 3, and the sum of their squares is 325. Find the numbers.

SIMILAR TRIANGLES

221. Triangles are called **similar** if they have the *same shape*.

Thus the triangles ABC and DBE of the figure are similar. Note that AB and DB have been divided into 7 and 3 equal parts, respectively. Hence the ratio of these sides is $\frac{7}{3}$.

What is the ratio of the sides BC and BE ? Of the sides CA and ED ? This is stated as follows: The lengths of the pairs of corresponding sides of two similar triangles form a proportion.



That is, we may write $\frac{AB}{DB} = \frac{CB}{EB} = \frac{CA}{ED}$.

Note that AB , BC , ... represent the **lengths** of these sides.

EXERCISES

1. If in two similar triangles the sides in one are 11, 13, and 16, and that side in the other which corresponds to the one whose length is 11 is 33, find the other sides of the second triangle.

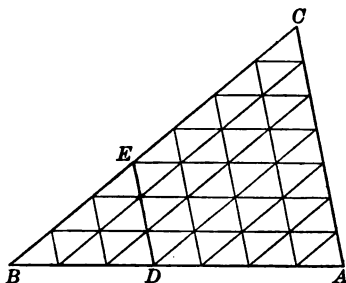
Solution. Let x represent the length of the side corresponding to the one whose length is 13. Then $\frac{x}{13} = \frac{33}{11}$, or $x = 39$.

In this manner find the remaining side.

2. If the sides of a triangle are 4, 6, and 10, and one side of a similar triangle is 9, find the remaining sides of the second triangle, if the given side corresponds to the side 4.

3. Solve Ex. 2 if the given side in the second triangle corresponds to 6.

4. Count the number of small triangles within each of the triangles ABC and DBE . Does the result illustrate the following statement:



The areas of similar triangles are in the same ratio as the squares of the lengths of their corresponding sides.

5. A triangular field has one side 15 rods long. Find the length of the corresponding side of a similar field whose area is 9 times as great.

Suggestion. The equation is $\frac{x^2}{(15)^2} = \frac{9}{1}$.

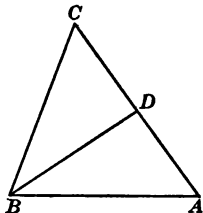
6. A triangular field has the sides 15, 18, and 27 rods, respectively. Find the dimensions of a similar field having 4 times the area.

7. The areas of two similar triangles are 49 and 64, respectively. One side of the first is 12. Find the corresponding side of the second.

8. A triangular field, one of whose sides is 20 rods, has an area of 80 square rods. Find the area of a similar field whose corresponding side is 45 rods.

It is found in geometry that if a line divides one angle of a triangle into two equal angles, it divides the opposite side into two parts which are in the same ratio as the other two sides of the triangle.

That is, in the figure $\frac{AD}{DC} = \frac{AB}{BC}$.



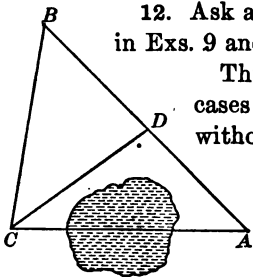
9. Knowing this fact, how many of the lines AD , DC , BC , and AB must you measure in order to find the rest of them?

10. If in the figure $AD = 8$, $DC = 6$, and $BC = 12$, find AB .

11. If in the figure $BC = 14$, $AB = 18$, and $AD = 6$, find DC .

12. Ask and answer two more questions like those in Exs. 9 and 10.

This fact about geometry enables us in some cases to find the distance between two points without measuring it directly.



13. If in the figure CD divides the angle at C into two equal parts and if you know the lengths of AD , DB , and BC , show fully how to find the length of a straight line across the pond from C to A without measuring it directly.

14. Suppose it is found that $BC = 75$ rods, $BD = 50$ rods and $AD = 25$ rods. Compute AC .

15. Two corresponding sides of two similar triangles are in the ratio 13 : 14. Show that the perimeters (sum of the sides) of the triangles are in the same ratio.

16. The perimeters of two similar triangles are in the ratio 33 : 35. Two sides of the first triangle are 8 and 12. Find two sides of the second triangle corresponding to the given sides of the first.

DRILL EXERCISES

Simplify:

$$1. \frac{x-1}{x-2} + \frac{x^2-3}{4-x^2} + \frac{x}{2+x}.$$

$$2. \frac{1}{(a-b)(b-c)} + \frac{1}{(c-b)(c-a)} - \frac{1}{(a-c)(b-a)}.$$

$$3. \left(a + 2b - \frac{3ab + 6b^2}{a+b}\right) \left(\frac{b^2 + a^2 + 2ab}{a^2 - 3ab - 4b^2}\right) \cdot \frac{1}{a+2b}$$

$$4. \left(a - b + \frac{4ab}{a-b}\right) + \left(\frac{(a+b)(b+2a)}{2a^2 + 3ab + b^2}\right).$$

$$5. \frac{\frac{a+b}{a-b} - \frac{a-b}{a+b}}{\frac{a^2+b^2}{a^2-b^2} - \frac{a^2-b^2}{a^2+b^2}}.$$

$$6. \frac{\frac{x-y}{x} - \frac{x+y}{y}}{\frac{x-y}{y} + \frac{x+y}{x}}.$$

$$7. \left\{ \frac{2}{x} - \frac{1}{a+x} + \frac{1}{a-x} \right\} + \left\{ \frac{a+x}{a-x} - \frac{a-x}{a+x} \right\}.$$

$$8. \left(2a - \frac{a^2-b^2}{a}\right) \left(3b + \frac{a^2+b^2}{b}\right) + \left(\frac{a^2}{b^2} + 5 + \frac{4b^2}{a^2}\right).$$

9. What is the value of $x^2 - 3x$ if $x = 1 + \sqrt{5}$? If $x = 1 - \sqrt{5}$?

10. What is the value of $3x^2 - 5x + 6$ if $x = \frac{2 - \sqrt{3}}{2}$? If $x = \frac{2 + \sqrt{3}}{2}$?

11. What is the value of $5x^2 + 7x$ if $x = \frac{a + \sqrt{b}}{2}$? If $x = \frac{a - \sqrt{b}}{2}$?

Solve and check by the method of § 198:

$$12. x^2 - 18x + 4 = 0.$$

$$15. 2x^2 - 7x = 5.$$

$$13. x^2 - 3ax + b = 0.$$

$$16. 3ax^2 - 7bx = 3.$$

$$14. x^2 + 9bx + c = 0.$$

$$17. 4a^2x^2 - 2bax = ca.$$

18. The edge of a cube is a inches long. Express in terms of a (1) its volume, (2) the total area of surface, (3) the length of diagonal of one face, and (4) the length of a diagonal of the cube.

CHAPTER XV

EQUATIONS INVOLVING FRACTIONS

EQUATIONS INVOLVING ALGEBRAIC FRACTIONS

222. We have already seen, § 101, that in solving an equation involving fractions the first step is to multiply both members of the equation by a number which will cancel all the denominators. Evidently, any common multiple of the denominators is such a multiplier. For the sake of simplicity, and for reasons which are fully discussed in the Advanced Course, the *lowest common multiple is always used for this purpose.*

223. **Illustrative Example.** Solve the following equation:

$$\frac{2x-1}{x-1} + \frac{4}{x+1} - \frac{3x}{x^2-1} = 2. \quad (1)$$

Solution. The L. C. M. of the denominators is $x^2 - 1$. In multiplying both members by $x^2 - 1$, we see that $x - 1$ is canceled in the first fraction, $x + 1$ in the second, and $x^2 - 1$ in the third,

$$\text{giving} \quad (2x-1)(x+1) + 4(x-1) - 3x = 2(x^2-1). \quad (2)$$

$$\text{Solving,} \quad 2x^2 + x - 1 + 4x - 4 - 3x = 2x^2 - 2. \quad (3)$$

$$2x = 3, \quad (4)$$

$$\text{and} \quad x = \frac{3}{2}. \quad (5)$$

Check by substituting $x = \frac{3}{2}$ in (1).

EXERCISES

Solve the following equations and check each solution by substituting in the original equation, except when the answer is given:

$$1. \quad \frac{3x-1}{x+1} - \frac{4x+3}{x-1} + \frac{x^2}{x^2-1} = -\frac{27}{x^2-1} + 1.$$

$$2. \frac{3x+5}{x-9} + \frac{2x+1}{x+2} = \frac{x-1}{x^2-7x-18}. \quad \text{Ans. } x = \frac{2}{3}, \text{ or } 1.$$

$$3. \frac{3x-4}{x+5} - \frac{4x-1}{x+4} + \frac{x^2+44}{x^2+9x+20} = 0.$$

$$4. \frac{x-4}{2x-10} - \frac{3x-15}{2x-6} = -\frac{3x^2-114}{4x^2-32x+60}.$$

$$\times 5. \frac{6(x+4)}{x+5} - \frac{3(2x-1)}{x+1} = \frac{7}{2}. \quad \text{Ans. } x = 1, \text{ or } -6\frac{1}{2}.$$

$$6. \frac{3x-4}{x-4} + \frac{5x-7}{2x-2} = \frac{9x^2-38}{2x^2-10x+8}. \quad \text{Ans. } x = 18\frac{1}{2}, \text{ or } 2.$$

$$7. \frac{x+17}{x+5} - \frac{2(x+6)}{x+3} = -\frac{x-1}{x+3}.$$

$$8. \frac{x+2}{x-5} + \frac{3x-15}{x-3} = \frac{3x-21}{x-3}.$$

$$9. \frac{2x-3}{-4x} + \frac{3x+1}{x-2} = \frac{4x+17}{x-2}. \quad \text{Ans. } x = \frac{-19 \pm \sqrt{345}}{4}.$$

$$10. \frac{3x-2}{2x+3} = \frac{2x^2+15x+28}{2x^2+5x+3} + \frac{2x-1}{x+1}.$$

$$\text{— } 11. \frac{2x-3}{2x+2} - \frac{x-8}{5x+2} = \frac{x+2}{2x+2}.$$

$$12. \frac{20x^2+7x-3}{9x^2-1} - \frac{3x+1}{3x-1} = 1.$$

$$13. \frac{7x^2+11x+4}{6x^2+13x+5} + \frac{x+3}{2x+1} = \frac{7x+11}{3x+5}.$$

$$14. \frac{3x+1}{5x-7} - \frac{x-3}{2x-7} = \frac{2x^2-10x+12}{10x^2-49x+49}.$$

224. Sometimes it is best to *add fractions before multiplying* by the L. C. M. and in other cases to multiply by the L. C. M. of *part of the denominators first*, and, after simplifying, multiply by the L. C. M. of the remaining denominators.

Ex. 1. Solve $\frac{1}{x-2} - \frac{1}{x-1} = \frac{1}{x-4} - \frac{1}{x-3}$. (1)

Adding fractions on the right and left,

$$\frac{1}{(x-2)(x-1)} = \frac{1}{(x-4)(x-3)}. \quad (2)$$

Multiplying by L. C. M., $(x-4)(x-3) = (x-2)(x-1)$. (3)

Hence, $x = 2\frac{1}{2}$. (4)

Check by substituting $x = 2\frac{1}{2}$ in equation (1).

Ex. 2. Solve $\frac{4t-3}{16} - \frac{t-2}{4} = \frac{2t-2}{5t+2}$. (1)

Multiplying by 16, $4t-3-4t+8 = \frac{32t-32}{5t+2}$. (2)

Hence, $5 = \frac{32t-32}{5t+2}$. (3)

Multiplying by $5t+2$, $25t+10 = 32t-32$. (4)

Hence, $t = 6$. (5)

Check by substituting $t = 6$ in equation (1).

In Exs. 1 and 2, try to solve by clearing of fractions completely at the outset and see why the plan of solving here given is simpler.

EXERCISES

1. $\frac{3x+6}{5} - \frac{9x+3}{15} = \frac{x+7}{6x-8} + 4$. Ans. $x = 1\frac{1}{5}$.

2. $\frac{7x+1}{12} - \frac{14x-22}{24} = \frac{11x+5}{8x-28}$.

3. $\frac{3x+4}{2} - \frac{12x+1}{8} = \frac{5x-1}{3x+2}$. Ans. $x = -7\frac{3}{8}$.

4. $\frac{7t+3}{5} - \frac{21t+9}{15} = \frac{17t-3}{3t+11} + 2$. Ans. $t = -1\frac{9}{11}$.

$$5. \frac{11v-15}{10} - \frac{33v+15}{30} = \frac{5v+5}{v-5}.$$

$$6. \frac{x-1}{x-2} + \frac{x-2}{3-x} = \frac{x-3}{x-4} - \frac{x-4}{x-5}.$$

$$7. \frac{1}{x-1} - \frac{2}{2x+1} = \frac{1}{x-2} - \frac{4}{4x+1}.$$

$$8. \frac{x-2}{x-3} - \frac{x-3}{x-4} = \frac{x-4}{x-5} + \frac{x-5}{6-x}.$$

$$\times 9. \frac{9}{x-7} - \frac{9}{x-2} = \frac{5}{x-8} - \frac{5}{x+1}.$$

$$10. \frac{x-1}{x-2} + \frac{x-2}{3-x} = \frac{x-3}{x-4} - \frac{x-5}{x-6}. \quad \text{Ans. } x = \pm 2\sqrt{3}.$$

SIMULTANEOUS FRACTIONAL EQUATIONS

225. When pairs of fractional equations are given, each should be reduced to the integral form before eliminating, except in special cases like those in the second following illustrative example.

Ex. 1. Solve the equations:

$$\left\{ \begin{array}{l} \frac{4}{x-y} + \frac{6}{x+y} = \frac{36}{x^2-y^2}. \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} \frac{3}{2x-y} - \frac{2}{x-3y} = \frac{-18}{(2x-y)(x-3y)}. \end{array} \right. \quad (2)$$

$$\text{From (1) by } M, \quad 4(x+y) + 6(x-y) = 36. \quad (3)$$

$$\text{By } F, D, \quad 5x - y = 18. \quad (4)$$

$$\text{From (2) by } M, \quad 3(x-3y) - 2(2x-y) = -18. \quad (5)$$

$$\text{By } F, D, \quad x + 7y = 18. \quad (6)$$

$$\text{From (4) by } M, \quad 35x - 7y = 126. \quad (7)$$

$$\text{Adding (6) and (7),} \quad 36x = 144. \quad (8)$$

$$\text{By } D, \quad x = 4. \quad (9)$$

$$\text{Substitute } x = 4 \text{ in (6),} \quad y = 2. \quad (10)$$

Check by substituting $x = 4, y = 2$ in (1) and (2).

Ex. 2. Solve the equations:

$$\begin{cases} \frac{2}{x} + \frac{3}{y} = 2, \end{cases} \quad (1)$$

$$\begin{cases} \frac{20}{x} - \frac{21}{y} = 3. \end{cases} \quad (2)$$

In this case it is best to solve the equations for $\frac{1}{x}$ and $\frac{1}{y}$ instead of for x and y .

From (1) by M ,
$$\frac{14}{x} + \frac{21}{y} = 14. \quad (3)$$

Adding (2) and (3),
$$\frac{34}{x} = 17. \quad (4)$$

Hence by D ,
$$\frac{1}{x} = \frac{1}{2}. \quad (5)$$

Substituting $\frac{1}{x} = \frac{1}{2}$ in (1),
$$\frac{1}{y} = \frac{1}{3}. \quad (6)$$

From (5) and (6) by M ,
$$x = 2, y = 3. \quad (7)$$

In Ex. 2 try to solve by first clearing of fractions and see why the plan here used is simpler.

EXERCISES

Solve the following equations:

$$\times \quad 1. \quad \begin{cases} \frac{2x-1}{x+1} - \frac{3y-1}{y+1} = \frac{-xy}{(x+1)(y+1)}, \\ \frac{x+2}{2y-1} + \frac{2x-1}{y+1} = \frac{5xy}{(2y-1)(y+1)}. \end{cases}$$

$$\vee \quad 2. \quad \begin{cases} \frac{3x+2}{3y-5} = \frac{x+1}{y-1}, \\ \frac{3x-2}{y+1} = \frac{3x-1}{y-1} - \frac{2}{(y-1)(y+1)}. \end{cases}$$

$$\begin{array}{lll} 3. \quad \begin{cases} \frac{5}{t} - \frac{6}{v} = 2, \\ \frac{17}{v} + \frac{4}{t} = 67. \end{cases} & 4. \quad \begin{cases} \frac{3}{x} + \frac{1}{y} = 21, \\ \frac{7}{x} - \frac{9}{y} = -19. \end{cases} & 5. \quad \begin{cases} \frac{7}{a} - \frac{1}{b} = 12\frac{1}{2}, \\ \frac{3}{a} + \frac{12}{b} = 24. \end{cases} \end{array}$$

$$6. \begin{cases} \frac{3}{x} - \frac{2}{y} = -4, \\ \frac{6}{x} + \frac{11}{y} = 52. \end{cases} \quad 7. \begin{cases} \frac{12}{x} - \frac{10}{y} = 1, \\ \frac{9}{x} + \frac{2}{y} = 15. \end{cases} \quad 8. \begin{cases} \frac{a}{x} + \frac{b}{y} = 1, \\ \frac{c}{x} + \frac{d}{y} = 1. \end{cases}$$

$$9. \begin{cases} \frac{1}{x} + \frac{1}{y} = 16, \\ \frac{1}{y} + \frac{1}{z} = 14, \\ \frac{1}{z} + \frac{1}{x} = 12. \end{cases} \quad 10. \begin{cases} \frac{1}{x} + \frac{1}{y} = a, \\ \frac{1}{y} + \frac{1}{z} = b, \\ \frac{1}{z} + \frac{1}{x} = c. \end{cases}$$

In Ex. 9 first add all three equations, and from half the sum subtract each equation separately. Likewise in Ex. 10.

PROBLEMS LEADING TO FRACTIONAL EQUATIONS

In solving the following problems use one or two unknowns as may be found most convenient.

1. There are two numbers whose sum is 51 such that if the greater is divided by their difference, the quotient is $3\frac{1}{2}$. Find the numbers.
2. Find two numbers whose sum is 91 such that if the greater is divided by their difference, the quotient is 7. Find the numbers.
3. There are two numbers whose sum is s such that if the greater is divided by their difference, the quotient is q . Find an expression in terms of s and q representing each number. Solve 1 and 2 by substituting in the formula just obtained.
4. What number must be subtracted from each term of the fraction $\frac{1}{\frac{1}{2}}$ so that the result shall be equal to $\frac{1}{2}$?
5. What number must be subtracted from each term of the fraction $\frac{2}{\frac{1}{3}}$ so that the result shall be equal to $\frac{2}{3}$?
6. What number must be subtracted from each term of the fraction $\frac{a}{b}$ so that the result shall be equal to $\frac{c}{d}$? Solve 4 and 5 by substituting in the formula obtained under 6.

7. What number must be added to each term of the fraction $\frac{1}{2}$ to obtain a fraction equal to $\frac{1}{4}$?

8. What number must be added to each term of the fraction $\frac{a}{b}$ to obtain a fraction equal to $\frac{c}{a}$?

9. There are two numbers whose difference is 153. If their sum is divided by the smaller, the quotient is equal to $\frac{4}{3}$.

10. There are two numbers whose difference is d . If their sum is divided by the smaller, the quotient is q . Find the numbers. Solve 9 by substituting in this formula.

11. Divide 548 into 2 parts, such that 7 times the first shall exceed 3 times the second by 474. *Ans.* One part is 211.8.

12. There are two numbers whose sum is 48 such that 3 times the first is 8 more than 5 times the second.

13. There are two numbers whose sum is s such that a times the first is b more than c times the second. Find both numbers.

14. What number must be subtracted from each of the numbers 12, 15, 19, and 25 in order that the remainders may form a proportion when taken in the order given?

15. What number must be added to each of the numbers 13, 21, 3, and 8 so that the sums shall be in proportion when taken in the order given?

16. What number must be added to each of the numbers a , b , c , d so that the sums shall be in proportion when taken in the given order?

17. What number must be subtracted from each of the numbers a , b , c , d so that the remainders shall be in proportion when taken in the given order?

Compare the results in 16 and 17 and explain the relation between them.

Solve Exs. 14 and 15 by substituting in the formulas obtained in 16 and 17.

18. There is a number composed of two digits whose sum is 11. If the number is divided by the difference between the digits, the quotient is $16\frac{1}{2}$. Find the number, the tens' digit being the larger.

19. There is a number composed of two digits whose sum is s . If the number is divided by the difference between the digits, the quotient is q . Find the number, the tens' digit being the larger.

20. **Illustrative Problem.** A can do a piece of work in 8 days, B can do it in 10 days. In how many days can they do it working together?

Since A can do the work in 8 days, in 1 day he can do $\frac{1}{8}$ of it, and since B can do it in 10 days, in 1 day he can do $\frac{1}{10}$ of it. If x is the number of days required when both work together, in 1 day they can do $\frac{1}{x}$ of it. Hence we have the equation,

$$\frac{1}{8} + \frac{1}{10} = \frac{1}{x}$$

21. A can do a piece of work in 12 days and B can do it in 9 days. How long will it take both working together to do it?

22. A pipe can fill a cistern in 11 hours and another in 13 hours. How long will it require both pipes to fill it? *Ans.* $5\frac{2}{11}$ hours.

23. A can do a piece of work in a days and B can do it in b days. How long will it take both together to do it?

24. A cistern can be filled by one pipe in 20 minutes and by another in 30 minutes. How long will it take to fill the cistern when both are running together?

25. A pipe can fill a cistern in 12 hours, another in 10 hours, and a third can empty it in 8 hours. How long will it require to fill the cistern when they are all running?

26. A man can do a piece of work in 18 days, another in 21 days, a third in 24 days, and a fourth in 10 days. How long will it require them when all are working together?

Ans. $4\frac{5}{17}$ days.

27. A and B working together can do a piece of work in 12 days. B and C working together can do it in 13 days, and A and C working together can do it in 10 days. How long will it require each to do it when working alone?

Suggestion: Let a = the fraction of the work A can do in one day, b = the fraction of the work B can do in one day, and c = the fraction of the work C can do in one day.

$$\text{Then,} \quad a + b = \frac{1}{12}, \quad b + c = \frac{1}{13}, \quad c + a = \frac{1}{10}.$$

28. A and B working together can do a piece of work in l days. B and C can do it in m days and C and A can do it in n days. How long will it require each working alone?

29. The circumference of the rear wheel of a carriage is 1.8 feet more than that of the front wheel. In running one mile the front wheel makes 48 revolutions more than the rear wheel. Find the circumference of each wheel.

If x is the number of feet in the circumference of the front wheel, then $\frac{5280}{x}$ is the number of revolutions in going one mile.

30. The circumference of the rear wheel of a carriage is 1 foot more than that of the front wheel. In going one mile the two wheels together make 920 revolutions. Find the circumference of each.

31. The distance from Chicago to Minneapolis is 420 miles. By increasing the speed of a certain train 7 miles per hour the running time is decreased by 2 hours. Find the speed of the train.

If r is the original rate of the train, then $\frac{420}{r}$ is the running time.

32. The distance from New York to Buffalo is 442 miles. By decreasing the speed of a fast freight 8 miles per hour the running time is increased 4 hours. Find the speed of the freight.

33. A motor boat goes 10 miles per hour in still water. In 10 hours the boat goes 42 miles up a river and back again. What is rate of the current?

DRILL EXERCISES

Approximate the square roots of each of the following to two places of decimals:

1. 7.9482.

3. 390.07.

5. .0048.

2. 4578.9.

4. 9.176.

6. .04791.

Solve the following equations:

7. $\sqrt{2x^2-2}=x-3$.

10. $\frac{\sqrt{5x+1}}{\sqrt{2x-3}} = \sqrt{\frac{6x-1}{3x-2}}$.

8. $\sqrt{2x+7} = \sqrt{x} + 3$.

11. $\sqrt{x+2} = 3 - \sqrt{3-x}$.

9. $\frac{\sqrt{a}}{\sqrt{x-a}} - \frac{\sqrt{x+2a}}{2\sqrt{a}} = 0$.

12. $\frac{9x+8}{\sqrt{8x+1}} = \sqrt{10x+19}$.

Simplify the following:

13. $\frac{x-y}{x-2z} - \frac{x+y}{x+2z} + \frac{y-4z^2}{4z^2-x^2}$.

14. $\left(4 - \frac{a^2+2ab+b^2}{ab}\right)\left(\frac{4ab}{a^2-2ab+b^2} + 1\right)$.

15. $\frac{a^3+27}{a^3-8} + \frac{a^2b-3ab+9b}{a^2+2a+4}$.

16. $\left(y + \frac{xy}{y-x}\right)\left(y - \frac{xy}{x+y}\right)\left(\frac{y^3-x^3}{y^2+x^2}\right)$.

17. $\left(\frac{x-2}{x-3} - \frac{x+3}{x+4}\right) + \left(\frac{1}{x+1} + \frac{7}{x-3}\right)$.

18. $\frac{\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}}{\frac{(a-b-c)(a+b+c)}{ab}}$.

20. $\frac{\frac{a}{b^2}}{1 + \frac{a^2}{b^2}} + \frac{\frac{a}{(b-a)^2}}{\frac{b+a}{b-a}}$.

19. $\frac{\left(a - \frac{3}{a}\right)^2 + 16}{\left(a + \frac{3}{a}\right)^2 + 4}$.

21. $\frac{2 + \frac{1}{x-2} + x}{x + \frac{1}{x+2} - 2}$.

CHAPTER XVI

GENERAL REVIEW

226. The purpose of this review is to reconsider some of the important topics of the course in order to show how they are *interrelated and unified by a few simple principles*, and to gain a little deeper insight into the *nature of algebraic processes and their uses*.

FIRST EXTENSION OF THE NUMBER SYSTEM

227. The number system of arithmetic consists of the integers including zero and the fractions of the form $\frac{m}{n}$, where m and n are integers (n not zero), together with certain indicated roots such as $\sqrt{2}$. In algebra we early encountered the *negative numbers*. These compelled us to give a distinct name to the numbers of arithmetic which were then called *positive numbers*. The principles according to which positive and negative numbers are added, subtracted, multiplied, and divided were studied in connection with concrete facts to which these numbers are naturally applicable.

EXERCISES

1. Do negative numbers apply to all things to which positive numbers apply? Can there be a negative number of books on a shelf?
2. Make a list of things to which both positive and negative numbers apply.
3. Show by concrete examples different from those in this book how Principle VII is obtained.

4. In arithmetic how did you find the difference in temperature between 8° above zero and 40° above zero? Between 8° above zero and 10° below zero? Note that in the latter case you *added* to find a *difference*. How does this compare with the subtraction of a negative number from a positive or a positive number from a negative? By concrete examples verify Principle VIII. Show how a problem in subtraction may be changed into a problem in addition by means of Principle VIII.

5. Verify Principle IX by concrete examples.

6. How may the signs of the factors of a product be changed without changing the value of the product? Make all possible changes of sign in $(a-b)(b-c)(c-d)$ which will not change its value. Also make all possible changes of sign which will change the sign of the product.

7. Define division in terms of multiplication. Derive Principle X from Principle IX by means of the definition of division.

8. State how the signs involved in a fraction may be changed without changing the value of the fraction. Make all possible changes of signs which will not change the value of $\frac{a}{b}$, also of $-\frac{a}{b}$.

9. Make all possible changes of signs which will leave unchanged the value of the expression $\frac{(x-y)(y-z)(z-x)}{(a-b)(b-c)}$.

10. Reduce $\frac{a-b}{x-y}$, $\frac{b-c}{y-z}$, and $\frac{a+b}{(y-x)(z-y)}$ to fractions having a common denominator.

11. Reduce $\frac{a}{(a-b)(b-c)}$, $\frac{b}{(c-d)(b-a)}$, and $\frac{c}{(c-b)(d-c)}$ to fractions having a common denominator.

12. Recall some problem solved during the year, in which a negative result, while satisfying the equation, does not have meaning in the problem.

228. Rational and Irrational Numbers. The positive and negative integers and fractions, including zero, are called **rational numbers**. In attempting to solve the equation $x^2 = 2$ we get $x = \pm \sqrt{2}$ in which $\sqrt{2}$ stands for a number whose square is 2. It may be shown that no rational number satisfies this condition. Such a number is called an **irrational number**.

In this book the only irrational numbers encountered are irrational square roots called quadratic surds, whose values we learned to approximate to any desired degree of accuracy. Other forms of irrational numbers are found in more advanced work.

The essential property of a quadratic surd is stated in principle XVIII.

SECOND EXTENSION OF THE NUMBER SYSTEM

229. The equation $x^2 + 1 = 0$ or $x^2 = -1$ introduces still another kind of number, namely, $x = \pm \sqrt{-1}$. Such numbers, called **imaginaries**, were met in the solution of quadratic equations, but their study was postponed to the Advanced Course.

The **irrational and imaginary numbers** form important topics in higher courses in mathematics.

The extension of the number system to include negative numbers and imaginaries, besides the rational and irrational numbers of arithmetic, is one of the chief distinctions between arithmetic and algebra.

LITERAL EQUATIONS

230. Identities. There are two essentially different kinds of literal equations. One is the *identity* of which an example is $a(b + c) = ab + ac$.

If all the indicated operations in an algebraic identity are performed, the two members become exactly alike. Hence the chief use of identities is to formulate those operations on algebraic expressions which *change their form but not their value*.

EXERCISES

1. What values of the letters involved in an identity will satisfy it?

2. State in the form of identities as many as possible of the eighteen principles given in this book.

Determine which of the following are identities:

3. $(x + y)^2 = x^2 + 2xy + y^2$.
5. $(x + 1)^2 = x^2 - 2$.
4. $(x - y)^2 = x^2 - 2xy + y^2$.
6. $a^2 + ab + b^2 = (a + b)^2 - ab$.
7. $a^2 - b^2 = (a - b)(a + b)$.
8. $(a - b)(a^2 + ab + b^2) = a^3 - b^3$.
9. $(a + b)(a^2 - ab + b^2) = a^3 + b^3$.
10. $(x + b)(x + a) = x^2 + (a + b)x + ab$.
11. $(x - a)(x + b) = x^2 - (a + b)x + ab$.
12. $(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.

231. **Equations of Condition.** The other kind of literal equation is one which is not satisfied for all values of all the letters involved. These originate in a large variety of ways.

For example in $p = br$, which may be regarded as a definition of percentage, any value may be assigned to two of the letters but the third is thereby determined. Likewise in $i = prt$, which may be regarded as a definition of interest, any value may be given to three of the letters thereby fixing the fourth.

In the equation $w_1d_1 = w_2d_2$, which is the law of the lever and was established by experimentation, any three letters may be given arbitrarily, thereby fixing the fourth.

If in a literal equation numerical values are assigned to all the letters except one, then the value of this one is determined by the equation.

232. In general, every literal equation which is not an identity states a certain relation that must exist between the variables involved. In this sense such an equation is an **equation of condition**.

In using a literal equation it may become necessary to solve it for any one of the letters in terms of the others, and we thus have an important example of the solution of equations in one unknown.

EXERCISES

1. State as many rules of arithmetic as you can in the form of literal equations and indicate which of these are identities and which are equations of condition.

2. Indicate which of the equations of the preceding example can be derived from others.

3. Enumerate the various steps which may be used in the solution of an equation in one unknown.

4. Solve for x and y the simultaneous equations

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2, \end{cases}$$

and reduce the results to the simplest form.

NOTE. — The letters a_1, a_2, b_1, b_2 are read a one, a two, b one, b two, etc. The numbers a_1 and a_2 , etc., are entirely distinct so far as their values are concerned, the subscripts 1 and 2 being used to distinguish *two different coefficients of the same letter x* .

In solving these equations the steps involved are exactly the same as are used in solving any other pair of linear equations. The result obtained may, therefore, be used as a formula from which the solution of any two such equations may be obtained.

5. Solve equations 1 to 5 on page 119 by means of this formula.

6. What equations would have to be solved to obtain a formula for the solution of three equations in three unknowns?

7. What statements can you make about the relation of the equation $ax^2 + bx + c = 0$ to any other quadratic equation?

VARIABLES, GRAPHS, FUNCTIONALITY

233. We have seen that an equation like $s = vt$, where v is a constant, expresses a fixed relation between the variables s and t .

Thus, if $v = 8$, then $s = 8t$, and the variables change subject to the functional relation expressed by this equation.

The study of the dependence of one variable on other variables is to-day engaging the best effort of great men in all the sciences. The attempt is constantly made in each case to so determine the nature of such dependence that it may be expressed in the form of a *literal equation*.

The cases of functional relations between variables which have been studied in this book are but a very few of the simplest kind and are confined to *two variables only*. The graph is a convenient means of exhibiting a known relation between two variables.

EXERCISES

1. Write an equation of the first degree in two variables x and y in which y increases as x increases.
2. Construct a graph representing the equation of the preceding.
3. Write a similar equation in which y decreases as x increases.
4. Construct a graph representing the equation obtained in the preceding.
5. How many points do you need to locate on the graph of a linear equation before you can draw the whole graph?
6. Construct graphs of the equations $3x - 2y = 12$ and $2x = 5y + 10$.
7. What is the relation between the graphs of the equations $2x + 3y = 6$ and $2x + 3y = 12$?

SIMULTANEOUS EQUATIONS

234. We have seen that one equation in two variables may be regarded as stating a relation between these variables subject to which they are permitted to vary.

Thus in the equation $x + y = 5$, x and y may vary at will so long as their sum remains 5.

Such an equation is satisfied by an endless number of pairs of values. While this is true of each of two equations in two variables, there is only *one pair* of values which satisfy both equations, provided the equations are independent and simultaneous and both of the first degree. Hence two linear equations, each in two variables, completely determine these variables. This is most easily seen by means of the graph. See page 112.

The solution of a pair of simultaneous linear equations consists in finding this one pair of numbers which satisfy both equations.

EXERCISES

1. By means of an example explain elimination by addition and subtraction.
2. By means of an example explain elimination by substitution.
3. Under what conditions is elimination by substitution simpler than elimination by addition and subtraction? Under what conditions is it not?

FACTORING, QUADRATICS AND FRACTIONS

235. To be able to select at sight the factors of simple algebraic expressions is a prerequisite for successful and effective algebraic manipulation. It will be well to review the various types of factoring given in the text. You should now be able to read at sight the factors of many of the exercises in Chapter IX, and to solve at sight most of the equations on page 168, after reducing each to the type form $ax^2 + bx + c = 0$.

EXERCISES

1. Make a list of the various forms of binomials which are factorable, and show how to find the factors.

2. Give a similar treatment of trinomials.

3. What forms of polynomials of four terms are you able to factor?

4. Can you factor the following polynomials of five terms:
(1) $a^2 + 2ab + b^2 + a + b$? (2) $4x^2 + 12xy + 9y^2 + 2x + 3y$?

5. In the exercises on pages 143-146 and 152-155 find as many as you can of the results at sight.

6. In the exercises on pages 228-232 find as many as you can of the results at sight. You should at least be able to write the results without first writing in full all the factors of each polynomial.

7. What is the form of the square of a polynomial. Explain the method of finding the square root of a polynomial and also of a number expressed in arabic figures.

236. Factors of Quadratic Expressions. We have seen that certain quadratic equations may be solved by means of factoring.

For example, $x^2 - 7x + 12 = 0$, in which $x - 3$ and $x - 4$ are the factors of the left member and 3 and 4 are the roots.

It is now possible to use the general solution of any quadratic equation $x^2 + px + q = 0$ for the purpose of factoring the trinomial $x^2 + px + q$.

In $x^2 - 7x + 12 = 0$ we observe that 4 is a root of the equation and that $x - 4$ is a factor of the left member. Also 3 is a root and $x - 3$ is a factor.

In like manner if r_1 and r_2 are the roots of the quadratic equation $x^2 + px + q = 0$, then $x - r_1$ and $x - r_2$ are the factors of $x^2 + px + q$.

EXAMPLE: Factor $x^2 + 2x - 5$.

Solution: Solving the equation $x^2 + 2x - 5 = 0$ we get $x = -1 + \sqrt{6}$ and $x = -1 - \sqrt{6}$. Hence the factors of $x^2 + 2x - 5$ are $x - (-1 + \sqrt{6})$ and $x - (-1 - \sqrt{6})$.

That is, $x^2 + 2x - 5 = (x + 1 - \sqrt{6})(x + 1 + \sqrt{6})$.

Verify by multiplying the factors together to see if the product is $x^2 + 2x - 5$. The multiplication of the factors thus found may be used as a check on the solution of the quadratic.

EXERCISES

Factor by the above method the following quadratic expressions and verify the results by multiplying the factors:

- | | | |
|-----------------------|---------------------|-----------------------|
| 1. $x^2 + 7x - 4$. | 4. $x^2 - 8x + 1$. | 7. $x^2 + ax - b$. |
| 2. $x^2 + 12x + 13$. | 5. $x^2 - 3x - 4$. | 8. $x^2 - 2bx - 5b$. |
| 3. $x^2 - 9x + 12$. | 6. $x^2 - ax + b$. | 9. $x^2 + 4ax + 2a$. |

OPERATIONS AT SIGHT

237. You should now be sufficiently familiar with the simple algebraic operations to carry out many of them without writing down each step. This is illustrated in the following examples.

Ex. 1. Solve $45x - 14 + 7x = 13x + 7 + 8x + 41$.

Solution: $31x = 62$ and $x = 2$.

We add $45x$ and $7x$ and then subtract $13x$ and $8x$ (the latter are to be subtracted from both members of the equation) getting $31x$. Then we add 14, 7, and 41 (14 must be added to each member of the equation) getting 62. All this is done *mentally*.

In this manner solve Equations 1 to 20, pages 26, 27.

Ex. 2. Add the polynomials $3ax + 5by - 8b^2c^2$, $4b^2c^2 - 2ax - 7by$, and $4ax - 2by + 7b^2c^2$.

Solution: We pick out at once all terms having the factor ax and add them, then the terms having the factor by , and finally the terms having the factor b^2c^2 obtaining $5ax - 4by + 3b^2c^2$.

Polynomials may be subtracted in the same way.

In this manner solve Exs. 1 to 18, page 61.

$$\text{Ex. 3. Solve } \begin{cases} 3x + 5y = 14, & (1) \\ 5x - 3y = 8. & (2) \end{cases}$$

Solution: $25y + 9y = 70 - 24 = 46$ and $y = 46 \div 34 = \frac{23}{17}$.

We notice that to eliminate x we must multiply equation (1) by 5 and equation (2) by 3 and then subtract. Since this will eliminate x , we need not write down the terms containing it. Hence we get $5 \times 5y - 3(-3y) = 5 \times 14 - 3 \times 8$, or $34y = 46$.

In this manner solve Exs. 10 to 20, page 117.

The following operations should, for the most part, be done at sight, or with very little writing:

- (1) Finding the products of any two binomials.
- (2) Practically all the factoring in this book.
- (3) Supplying the third term to complete any trinomial square, when two terms are given.
- (4) Solving a quadratic by means of the formula.
- (5) Most of the cancellations needed in dealing with fractions.
- (6) Selection of the rationalizing factor for a quadratic surd and applying it.

Turn now to these various topics and consider each exercise a *challenge* to your mental skill.

Such a familiarity with the operations of algebra as is here indicated will greatly aid you, not only in your future work in mathematics, but also in the study of a science such as physics.

PROBLEMS

238. The solution of problems by means of equations is an important feature of algebra. It is a much more powerful method than arithmetic provides. Even though the operations be actually the same in both cases, yet the *equation* enables us to tabulate the data of a problem in such a way that they can be handled clearly and concisely and so that we may obtain the solution by means of a few standard operations.

239. Problems have in all ages held a peculiar interest for the human mind. Some are of interest because of the challenge they present to the mind as mere puzzles. Such is the following:

Ex. 1. A father is 9 years less than twice as old as his son and 12 years ago he was 19 years less than three times as old as the son. Find the age of each.

Other problems stated in the puzzle form attract attention because of the interesting facts revealed by their solutions. For example:

Ex. 2. The Fahrenheit reading at the temperature of liquid air is 128 degrees lower than the Centigrade reading. Find both the Centigrade and the Fahrenheit reading at this temperature.

Still other problems, while mere puzzles, are of value because they deal with important scientific or mathematical data. For example:

Ex. 3. A boatman trying to row up a river drifts back at the rate of $1\frac{1}{2}$ miles per hour, while he can row down the river at the rate of 12 miles per hour. What is the rate of the current?

Ex. 4. A beam is 12 feet long. It carries a 40-pound weight at one end, a 60-pound weight 3 feet from this end, and a 70-pound weight at the other end. Where is the fulcrum if the beam is balanced?

Ex. 5. There is a rectangle whose length is 60 feet more, and whose width is 20 feet less, than the side of a square of equal area. Find the dimensions of the square and the rectangle.

But by far the most important problems are those which it is necessary for mankind to solve in order to gain certain desired information. Such problems are called **real problems** or **practical problems**. For example:

Ex. 6. A farmer has a field 120 rods long and 60 rods wide. How wide a strip must he plow around it to make 5 acres?

Ex. 7. Find the average temperature for 12 hours from the following hourly observations; -10° ; -7° ; -5° ; -5° ; -1° ; $+3^{\circ}$; $+7^{\circ}$; $+9^{\circ}$; $+9^{\circ}$; $+4^{\circ}$; $+2^{\circ}$; -4° .

Ex. 8. The earth and Mars were in conjunction. When are they next in conjunction if the earth's period is 365 days and that of Mars 687 days? (See figure, page 269.)

240. The development of every physical science depends to a large extent upon the solution of problems of this latter type, and algebra is one of the great tools for this purpose. It is not possible for you at this stage to apply algebra to the solution of many kinds of problems of this type because of your limited knowledge of the scientific principles involved in their statement. But if you have mastered the various processes of solving equations, if you are familiar with factoring and fractions, with square root and radicals, in short, with all the processes used in this course, you are thus in possession of the tools with which to solve any problems whose conditions you can translate into statements in the form of linear or quadratic equations.

SUPPLEMENTARY PROBLEMS ON CHAPTER II

1. The area of Louisiana is (nearly) 4 times that of Maryland, and the sum of their areas is 60,930 square miles. Find the (approximate) area of each state.

2. The horse power of a certain steam yacht is 12 times that of a motor boat. The sum of their horse-powers is 195. Find the horse power of each.

3. At a football game there were 2000 persons. The number of women was 3 times the number of children, and the number of men was 6 times the number of children. How many men, women, and children were there?

4. It is twice as far from New York to Syracuse as from New York to Albany, and it is 4 times as far from New York to Cleveland as from New York to Albany. The sum of the three distances is 1015 miles. Find each distance.

5. The altitude of Popocatepetl is 1716 feet less than that of Mt. Logan, and the altitude of Mt. St. Elias is 316 feet greater than that of Popocatepetl. Find the altitude of each mountain, the sum of their altitudes being 55,384 feet.

6. It is 4 times as far from New York City to Cincinnati as from New York to Baltimore. Twice the distance from New York to Cincinnati minus 5 times that from New York to Baltimore equals 567 miles. How far is it from New York to each of the other cities?

7. The melting temperature of glass is 276 degrees (Centigrade) higher than twice that of zinc. One-half the number of degrees at which glass melts plus 7 times the number at which zinc melts equals 3434. Find the melting point of each.

8. The melting temperature of nickel is 496 degrees (Centigrade) higher than that of silver. Three times the number of degrees at which nickel melts plus 2 times the number at which silver melts equals 6258. Find the melting point of each.

9. A cubic foot of nickel weighs 80 pounds more than one cubic foot of tin. Four cubic feet of nickel plus 2 cubic feet of tin weigh 3056 pounds. Find the weight per cubic foot of each metal.

10. A cubic foot of gold weighs 545 pounds more than a cubic foot of silver. One cubic foot of gold and one cubic foot of silver together weigh 1855 pounds. Find the weight per cubic foot of each metal.

11. A cubic foot of steel weighs 17 times as much as a cubic foot of yellow pine. The combined weight of 11 cubic feet of pine and 3 cubic feet of steel is 1773.2 pounds. Find the weight of one cubic foot of each.

12. The area of Great Britain is 7557 square miles more than 9 times that of the Netherlands, and the area of Japan is 42,065 square miles less than 15 times that of the Netherlands. One-third the area of Great Britain plus $\frac{1}{2}$ the area of Japan is 69,994 square miles. Find the area of each country.

13. The diameter of the earth is 1918 miles more than twice that of Mercury, and the diameter of Venus is 1700 miles more than twice that of Mercury. The diameter of the earth plus $\frac{1}{2}$ that of Venus equals 11,768 miles. Find the diameter of each planet.

14. The diameter of Jupiter is 500 miles more than 20 times that of Mars, and the diameter of Saturn is 4200 miles more than 16 times that of Mars. One-tenth the diameter of Jupiter plus $\frac{1}{2}$ that of Saturn is 45,150 miles. Find the diameter of each planet.

15. The diameter of Neptune is 2900 miles more than that of Uranus. The sum of their diameters is 66,700 miles. Find the diameter of each planet.

16. The money circulation of the United States in 1880 was 136 million dollars more than 3 times that in 1850. In 1910 it was 51 million more than 11 times that in 1850. The circulation of 1910 exceeded that of 1880 by 2147 million. Find the circulation in each year.

SUPPLEMENTARY PROBLEMS ON CHAPTER VI

1. A sparrow flies 135 feet per second and a hawk 149 feet per second. The hawk in pursuing the sparrow passes a certain point 7 seconds after the sparrow. In how many seconds from this time does the hawk overtake the sparrow?

2. A courier starts from a certain point, traveling v_1 miles per hour, and a hours later a second courier starts, going at the rate of v_2 miles per hour. In how long a time will the second overtake the first, supposing v_2 greater than v_1 ?

If the second courier requires t hours to overtake the first, the latter had been on the way $t + a$ hours. Thus the distance covered by the second courier is $v_2 t$ and by the first $v_1(t + a)$. As these numbers are equal we have

$$v_2 t = v_1(t + a).$$

This formula summarizes the solution of all problems like 3 and 4 below.

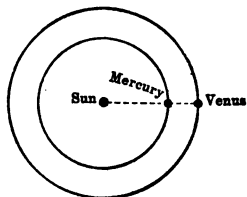
3. In an automobile race A drives his machine at an average rate of 53 miles per hour, while B , who starts $\frac{1}{4}$ hour later, averages 57 miles per hour. How long does it require B to overtake A ? Use the above formula.

4. A freight steamer leaves New York for Liverpool, averaging $10\frac{1}{2}$ knots per hour, and is followed 4 days later by an ocean greyhound, averaging $25\frac{1}{2}$ knots per hour. In how long a time will the latter overtake the former?

5. One athlete makes a lap on an oval track in 26 seconds, another in 28 seconds. If they start together in the same direction, in how many seconds will the first gain one lap on the other? Two laps?

Let one lap be the unit of distance. Since the first covers one lap in 26 seconds, his rate per second is $\frac{1}{26}$. Likewise the rate of the other is $\frac{1}{28}$. If t is the required number of seconds, the distance covered by the first is $\frac{1}{26}t$ and by the second $\frac{1}{28}t$. If the first goes one lap farther than the second, the equation is $\frac{1}{26}t = \frac{1}{28}t + 1$.

6. Two automobiles are racing on a circular track. One makes the circuit in 31 minutes and the other in $38\frac{1}{2}$ minutes. In what time will the faster machine gain 1 lap on the slower?



7. The planet Mercury makes a circuit around the sun in 3 months and Venus in $7\frac{1}{2}$ months. Starting in conjunction, as in the figure, how long before they will again be in this position?

Note that the problem may be solved the same as if the two planets were moving in the same orbit at different rates.

8. Saturn goes around the sun in 29 years and Jupiter in 12 years. Starting in conjunction, how soon will they be in conjunction again?

9. Uranus makes the circuit of its orbit in 84 years and Neptune in 164 years. If they start in conjunction, how long before they will be in conjunction again? *Ans.* $172\frac{1}{2}$ years.

10. The hour hand of a watch makes one revolution in 12 hours and the minute hand in 1 hour. How long is it from the time when the hands are together until they are again together?

11. One object makes a complete circuit in a units of time and another in b units (of the same kind). In how many units of time will one overtake the other, supposing b to be greater than a ?

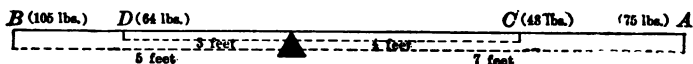
The solution of this problem summarizes the solution of all problems like those from 5 to 10.

12. At what times between 12 o'clock and 6 o'clock are the hands of a watch together? (Find the time required to gain one circuit, two circuits, etc.)

PROBLEMS INVOLVING THE LEVER

1. A teeter board is in balance when two boys, A and B, weighing 105 and 75 pounds, respectively, are seated at distances 5 and 7 feet from the fulcrum because $7 \cdot 75 = 5 \cdot 105$. If now two boys weighing 48 and 64 pounds are seated on the same board with the other boys, the teeter will again be in balance if their distances are 4 and 3 feet, because

$$7 \cdot 75 + 4 \cdot 48 = 5 \cdot 105 + 3 \cdot 64$$



The weight of the boy multiplied by his distance from the fulcrum is called his **leverage**. The sum of the leverages on the two sides must be the same. Hence, if two boys, weighing respectively, w_1 and w_2 pounds, are sitting at distances d_1 and d_2 on one side, and two boys, weighing w_3 and w_4 pounds, sitting at distances d_3 , d_4 on the other side, then

$$w_1 d_1 + w_2 d_2 = w_3 d_3 + w_4 d_4. \quad (2)$$

2. If two boys weighing 75 and 90 pounds sit at distances of 3 and 5 feet respectively on one side and one weighing

82 pounds sits at 3 feet on the other side, where should a boy weighing 100 pounds sit to make the board balance?

3. A beam carries a weight of 240 pounds $7\frac{1}{2}$ feet from the fulcrum and a weight of 265 pounds at the opposite end which is 10 feet from the fulcrum. On which side and how far from the fulcrum should a weight of 170 pounds be placed so as to make the beam balance?

4. Two boys, *A* and *B*, having a 50-lb. weight and a teeter board, proceed to determine their respective weights as follows: They find that they balance when *B* is 9 feet and *A* 7 feet from the fulcrum. If *B* places the 50-lb. weight on the board beside him, they balance when *B* is 3 and *A* 4 feet from the fulcrum. How heavy is each boy?

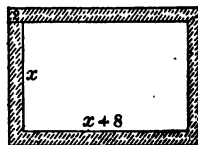
5. *C* is $6\frac{1}{2}$ feet from the point of support and balances *D*, who is at an unknown distance from this point. *C* places a 33-lb. weight beside himself on the board and, when $4\frac{2}{3}$ feet from the fulcrum, balances *D* who remains at the same point as before. *D*'s weight is 84 pounds. What is *C*'s weight, and how far is *D* from the fulcrum?

6. *E* weighs 95 pounds and *F* 110 pounds. They balance at certain unknown distances from the fulcrum. *E* then takes a 30-pound weight on the board, which compels *F* to move 3 feet farther from the fulcrum. How far from the fulcrum was each of the boys at first?

PROBLEMS INVOLVING GEOMETRY

1. A picture is 4 inches longer than it is wide. Another picture, which is 12 inches longer and 6 inches narrower, contains the same number of square inches. Find the dimensions of the pictures.

2. A picture, not including the frame, is 8 inches longer than it is wide. The area of the frame, which is 2 inches wide, is 176 square inches. Find the dimensions of the picture.

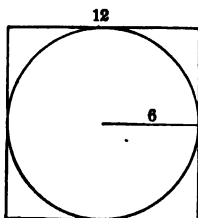


3. A picture, including the frame, is 10 inches longer than it is wide. The area of the frame, which is 3 inches wide, is 192 square inches. What are the dimensions of the picture?

4. The base of a triangle is 11 inches greater than its altitude. If the altitude and base are both decreased 7 inches, the area is decreased 119 square inches. Find the base and altitude of the triangle.

5. The base of a triangle is 3 inches less than its altitude. If the altitude and base are both increased by 5 inches, the area is increased by 155 square inches. Find the base and altitude of the triangle.

6. A square is inscribed in a circle and another circumscribed about it. The area of the strip inclosed by the two squares is 25 square inches. Find the radius of the circle.



7. Find the sum of the areas of a circle of radius 6 and the square circumscribed about the circle.

The area of the circle is $6^2 \pi = 36 \pi$, and the area of the square is $4 \cdot 6^2 = 4 \cdot 36$; i.e. the square contains 4 squares whose sides are 6. The sum of the areas is

$$4 \cdot 36 + 36 \pi = (4 + \pi) 36 = (4 + 3\frac{1}{2}) 36.$$

8. Find an expression for the sum of the areas of a circle of radius r and the circumscribed square. (Solve 7 by substituting in the formula here obtained.)

9. If the sum of the areas of a circle and the circumscribed square is 64, find the radius of the circle.

By the formula obtained under Ex. 8,

$$64 = (4 + \pi) r^2 = 3\frac{1}{2} r^2.$$

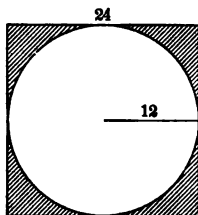
Hence,

$$r = \sqrt{8.96} = 2.99.$$

10. If the sum of the areas of a circle and the circumscribed square is 640 square feet, find the radius of the circle.

11. The sum of the areas of a circle and the circumscribed square is a . Find an expression representing the radius of the circle. (Replace π by $3\frac{1}{2}$ before simplifying.)

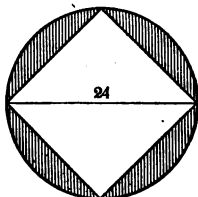
12. If the radius of a circle is 12, find the difference between the areas of the circle and the circumscribed square.



13. If the radius of a circle is r , find the difference between the areas of the circle and the circumscribed square. (Solve Ex. 12 by the use of the formula obtained here.)

14. If the radius of a circle is 16, find the area of the inscribed square. (This is the same problem as finding the area of a square whose diagonal is 32. See problems 8 and 9, page 190.)

15. If the radius of a circle is r , find an expression representing the area of the inscribed square. (This is problem 9, page 190.)

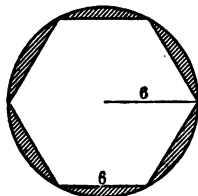


16. If the radius of a circle is 12, find the difference between the area of the circle and the area of the inscribed square.

17. If the radius of a circle is r , find an expression representing the difference between the areas of the circle and the inscribed square.

18. The radius of a circle is 10. Find the area of an inscribed hexagon. (See note, page 192.)

19. The radius of a circle is 6. Find the difference between the areas of the circle and the inscribed hexagon.



20. Find an expression representing the difference between the areas of a circle with radius r and the inscribed regular hexagon.

MISCELLANEOUS PROBLEMS

1. Divide the number 645 into two parts, such that 13 times the first part is 20 more than 6 times the other.

2. Divide the number a into two parts, such that b times the first part is c more than d times the second part.

3. The sum of three numbers is 98. The second is 7 greater than the first, and the third is 9 greater than the second. What are the numbers?

4. The sum of three numbers is s . The second is a greater than the first, and the third is b greater than the second. What are the numbers?

5. One boy runs around a circular track in 26 seconds, and another in 30 seconds. In how many seconds will they again be together, if they start at the same time and place and run in the same direction?

6. A bird flying with the wind goes 65 miles per hour, and flying against a wind twice as strong it goes 20 miles per hour. What is the rate of the wind in each case?

7. A steamer going with the tide makes 19 miles per hour, and going against a current $\frac{1}{2}$ as strong it makes 13 miles per hour. What is the speed of the steamer in still water?

8. Find the time between 4 and 5 o'clock when the hands of the clock are 30 minute spaces apart.

9. A man takes out a life insurance policy for which he pays in a single payment. Thirteen years later he dies and the company pays \$12,600 to his estate. It was found that his investment yielded 2% simple interest. How much did he pay for the policy?

10. After deducting a commission of 3% for selling bonds, a broker forwarded \$824.50. What was the selling price of the bonds?

11. A broker sold stocks for \$1728 and remitted \$1693.44 to his employer. What was the rate of his commission?

12. The difference between the areas of a circle and its circumscribed square is 12 square inches. Find the radius of the circle. (See problem 9, page 272.)

13. The difference between the areas of a circle and its inscribed square is 12 square inches. Find the radius of the circle.

14. The difference between the areas of a circle and the regular inscribed hexagon is 12 square inches. Find the radius of the circle.

15. The altitude of an equilateral triangle is 6. Find its side and also its area. Find the side and area, if the altitude is h .

16. The radius of a circle is 3 feet. Find the area of the regular circumscribed hexagon. Find the area if the radius is r feet.

17. The radius of a circle is r . Find the difference between the areas of the circle and the regular circumscribed hexagon.

18. The difference between the areas of a circle and the regular circumscribed hexagon is 9 square inches. Find the radius of the circle.

19. A circle is inscribed in a square and another circumscribed about it. The area of the ring formed by the two circles is 25 square inches. How long is the side of the square?

20. In a building there are at work 18 carpenters, 7 plumbers, 13 plasterers, and 6 hod carriers. Each plasterer gets \$1.90 per day more than the hod carriers, the carpenters get 35 cents per day more than the plasterers, and the plumbers 50 cents per day more than the carpenters. If one day's wages of all the men amount to \$183.45, how much does each get per day?

21. A train running 46 miles per hour leaves Chicago for New York at 7 A.M. Another train running 56 miles per hour leaves at 9.30 A.M. Find when the trains will be 15 miles apart. (Two answers.)

22. There is a number consisting of three digits, those in tens' and units' places being the same. The digit in hundreds' place is 4 times that in units' place. If the order of the digits is reversed, the number is decreased by 594. What is the number?

23. A hound pursuing a deer gains 400 yards in 25 minutes. If the deer runs 1300 yards a minute, how fast does the hound run? If the hound gains v_1 yards in t minutes and the deer runs v_2 yards per minute, find the speed of the hound.

24. A disabled steamer 240 knots from port is making only 4 knots an hour. By wireless telegraphy she signals a tug, which comes out to meet her at 17 knots an hour. In how long a time will they meet? If the steamer is s knots from port and making v_1 knots per hour, and if the tug makes v_2 knots per hour, find how long before they will meet.

25. A motor boat starts $7\frac{1}{2}$ miles behind a sailboat and runs 11 miles per hour while the sailboat makes $6\frac{1}{2}$ miles per hour. How far apart will they be after sailing $1\frac{1}{2}$ hours? If the motor boat starts s miles behind the sailboat and runs v_1 miles per hour, while the sailboat runs v_2 miles per hour, how far apart will they be in t hours?

26. An ocean liner making 21 knots an hour leaves port when a freight boat making 8 knots an hour is already 1240 knots out. In how long a time will the two boats be 280 knots apart? Is there more than one such position? If the liner makes v_1 knots per hour and the freight boat, which is s_1 knots out, makes v_2 knots per hour, how long before they will be s_2 knots apart?

27. A passenger train running 45 miles per hour leaves one terminal of a railroad at the same time that a freight running 18 miles per hour leaves the other. If the distance is 500 miles, in how many hours will they meet? If they meet in 8 hours, how long is the road? If the rates of the trains are v_1 and v_2 , and the road is s miles long, find the time.

INDEX

[References are to pages.]

Abscissa	109	Coördinates	109
Absolute value	37	Cross products	156
Addition	5, 39, 238	Cubes, sum or difference of	152, 153
by counting	38	Degree of an equation	111, 128, 166
of fractions	83, 224	Difference	6, 42
of polynomials	57	of two cubes	153
principles of	6, 57	of two squares	144
of radicals	186	Digit problems	65
of signed numbers	38	Directions for written work	25
Algebraic expressions	31	Distributive law	11
fractions	215	Dividend, divisor	49
operations	4	Division	48
sum	39	of fractions	86, 228
Alternation	238	indicated	2, 81
Antecedent	236	by a monomial	139, 140
Approximate roots	182	by a polynomial	148
Area problems	191, 192	principles of	10, 13, 49, 138
Associative law	57, 67	of a product	13
Averages of signed numbers	41	of signed numbers	48
Axes of coördinates	109	of a sum or difference	10
Base of a power	134	by zero	24, 149
Binomial	56	Drill Exercises	16, 55, 66, 80, 91, 101, 115, 123, 133, 147, 165, 175, 188, 198, 206, 214, 227, 235, 244, 254
Brace, bracket, etc.	7	Elimination, by addition or sub-	
Checking results	6, 28, 149, 202, etc.	traction	116
Clearing of fractions	88, 245	by substitution	118
Coefficient	5	Equality	7
Common denominator	82, 220	Equations	20
factor	5, 215	conditional	20, 258
multiple	82, 217	degree of	111, 128, 166
Commutative law	57, 67	graphic representation of	108
Completing the square	199	graphic solution of	112
Hindu method	201	identical	20, 257
Complex fraction	233		
Conditional equation	20, 258		
Consequent	236		

[References are to pages.]

- Equations** (*continued*) —
- independent 113
 - involving radicals 196
 - irrational 196
 - linear 111
 - literal 92
 - quadratic 166, 171, 198
 - root of 21
 - simultaneous 113, 116, 129
 - solution of 24, 113
 - solved by factoring 166
 - translation of 39
- Exponents** 67
- laws of 136, 138
 - negative, fractional 139
 - of a power 134
 - zero 139
- Expression, algebraic** 4, 31, 56
- fractional 81, 183, 215
 - rational, irrational 257
- Extremes, of a proportion** 236
- Factoring** 134, 261
- equations solved by 166
- Factors** 5, 134
- found by grouping 158
 - found by the quadratic formula 262
 - highest common 215
 - summary of 161
- Form changes** 23
- Formulas** 1, 3, 92, 203
- Fourth proportional** 240
- Fractional equations** 87, 245
- expressions 81, 183, 215
- Fractions** 81, 215
- clearing of 88, 245
 - complex 233
 - reduction of 219, 220
 - signs of 221
- Functional relation** 111, 260
- Fundamental laws** 11, 57, 67
- Graphic representation** 37, 102
- of motion 104
 - for solution of equations 108, 113
 - for solution of problems 106
- Graphs, of linear equations** 110, 260
- of statistics 102
- Grouping terms in factoring** 158
- Highest common factor** 215
- Historical Notes** 7, 11, 19, 50, 57, 114
- Identities** 7, 22, 257
- Imaginary numbers** 204, 257
- Index** 134
- Integers** 37
- Interest problems** 3, 29, 92
- Inversion of a proportion** 238
- Irrational numbers** 193, 257
- Laws of exponents** 136, 138
- Lever problems** 98, 271
- Like terms** 57, 58
- Linear equations** 111, 116
- Linear and quadratic** 171, 207
- Literal equations** 1, 3, 92, 122, 256
- fractions 215
 - problems 93, 191
- Long division** 148
- Lowest common multiple** 82, 221
- Lowest terms of a fraction** 219
- Mean proportional** 240
- Means, of a proportion** 236
- Members of an equality** 7
- Minuend** 42
- Monomial** 56
- Motion problems** 95, 268, 275
- Multiple** 82, 217
- lowest common 217
- Multiplication** 9, 12
- of fractions 86, 228
 - of polynomials 67
 - of radicals 194
 - of signed numbers 46
- Negative, exponent** 139
- number 36, 50
 - results, interpretation of 51

[References are to pages.]

- Number, absolute value of** . . . 37
 constant, variable . . . 111
 expression . . . 4, 56
 positive and negative . . . 36
 rational, irrational . . . 193, 257
 unknown . . . 21, 116, 124
Number system, of arithmetic 37
 of algebra . . . 37, 255, 257
Operations . . . 2
 at sight . . . 263
 order of . . . 15
Ordinates . . . 109
Parentheses . . . 7
 removal of . . . 62
Polynomials . . . 56
 addition and subtraction of . . . 57
 multiplication of . . . 67
 square root of . . . 178
Positive number . . . 36, 50
Power . . . 134
Prime factor . . . 218
Principles, eighteen fundamental,
 see *List* on page vii.
 importance of . . . 17, 67
Problems, applying theory of
 radicals . . . 189
 classified by topics 65, 73, 95, 204
 emphasizing use of formulas 93,
 189, 259, 269
 involving data of intrinsic
 interest 33, 131, 191, 243, 266
 solved by means of graphs . . . 107
 solved by factoring . . . 169
Products, of fractions . . . 86, 228
 special . . . 74, 134
Proportion . . . 236
Pythagorean proposition . . . 172
Quadratic equations 166, 171, 199
 properties of roots of . . . 202
 simultaneous with linear 171, 207
 solution by completing the
 square . . . 199, 200, 201
 solution by factoring . . . 166
 solution by formula . . . 203
Quotients, of fractions . . . 86, 228
 of polynomials . . . 148
 special . . . 152, 153
Radical expressions . . . 185, 193
Radicals . . . 176, 184, 193
 applications of . . . 189, 193
 equations involving . . . 196
 simplification of . . . 185, 193
Ratio . . . 236
Rational numbers . . . 193, 257
Rationalizing denominators 196
Remainder . . . 42
Reviews 17, 35, 53, 79, 90, 100, 114,
 132, 164, 197, 213, 255
Roots and powers . . . 134, 142
 square . . . 142, 176
Roots of equations . . . 21
Scale of signed numbers . . . 37
Signed numbers . . . 37, 45
Signs, of aggregation . . . 7
 of operation . . . 2
 of quality . . . 37
Similar, radicals . . . 186
 terms . . . 57
 triangles . . . 241
Simultaneous equations
 113, 116, 128
 fractional . . . 248
 linear . . . 113, 116
 linear and quadratic . . . 171, 207
Solution of equations 20, 113, 261
 by formula . . . 203
 by graphs . . . 113
Solution of problems . . . 27
 classified list . . . 93, 264
 hints on . . . 27, 31
Square, of a binomial . . . 74
 of a polynomial . . . 159
Square root . . . 142, 159, 176
 applications of . . . 189

[References are to pages.]

Square root (<i>continued</i>) —		Systems of equations	
approximate	182	113, 116, 128, 171, 207, 261	
of fractions	183	Terms, compound	56
of monomials	177	of a fraction	81, 219
of polynomials	178	of a proportion	236
Substitution, elimination by .	118	simple	56
Subtraction	5, 42	Trinomial	56
of fractions	83, 224	Trinomial squares	142
of polynomials	57		
of radicals	186	Unknown numbers	21, 116, 129
of signed numbers	42	Variables	111, 124, 128, 260
Sum	6, 38, 186	Variation, direct	236
of two cubes	152		
Surd	193	Written work, directions for .	25
quadratic	193		
Symbols, of aggregation . . .	7	Zero, division by	24, 149
of operation	3	multiplication by	24





